In Mem Edward Bright
Math. Dept.

SELECTED TOPICS IN THE THEORY OF DIVERGENT SERIES AND OF CONTINUED FRACTIONS.

BY EDWARD B. VAN VLECK.

PART I.

LECTURES 1–4. DIVERGENT SERIES.

IT may not be inappropriate for me to preface the first four lectures with a few words of a general character concerning divergent series. These will serve the double purpose of indicating the nature of the problems to be treated and of binding together the separate lectures.

The problem presented by any divergent series is essentially a functional one. When a divergent series of numbers is given, its genesis is usually to be found in some known or unknown function. The value which we attach to it is defined as the limit of a suitably chosen convergent process, and the elements of the process are the terms of the given series or are functions having these terms for their individual limits. Most commonly the given numerical series

$$a_0 + a_1 + a_2 + \cdots$$

is connected with the power series

$$(1) \qquad a_0 + a_1x + a_2x^2 + \cdots,$$

and the question thus reduces to that of determining under what conditions or restrictions a value may be assigned to the latter series when x approaches 1. The primary topic therefore is the divergent power series, and to this we shall confine our attention exclusively.

This topic, if broadly considered, presents itself under at least four very different aspects. What is given is in every case a power series with a radius of convergence which is not infinite. Suppose first that the radius is greater than zero and that the circle of convergence is not a natural boundary. Then the series defines within this circle an analytic function. In the region of divergence without the circle the value of the function may be obtained by the familiar process of analytic continuation. Theoretically the determination of the function is a satisfactory one, for *Poincaré* * has shown that the function throughout the domain in which it is regular can be obtained by means of an enumerable set of elements, $P_n(x - a_n)$. Practically, however, when *Weierstrass'* process is employed for analytic continuation, the labor is so excessive as to render the process nearly valueless except for purposes of definition. Hence to-day a search is being made for a workable substitute. I may refer particularly in this connection to the investigations by *Borel* and *Mittag-Leffler*. As I consider the work of the former to be both suggestive and practical, I have taken it as the basis of my second lecture.

A second aspect of our topic, intimately connected with the continuation of the function defined by (1), is the determination of the position and character of its singularities in the region where the series diverges. This subject is treated in Lecture 3.

When the circle of convergence is a natural boundary, it does not appear to be impossible, despite the earlier view of *Poincaré* to the contrary,† to discover, at least in a certain class of cases, an appropriate, although a non-analytic mode of continuing the function across the boundary into other regions where it will be again analytic. The thesis of *Borel* and its recent continuation in the *Acta Mathematica*, together with some excellent remarks by *Fabry*,‡ appear to be about all that has been done in this direction. A very brief discussion of the subject will be given in the fourth lecture in connection with series of polynomials and of rational fractions.

Lastly, we have the conundrum of the truly divergent power series — the series which converges only when $x = 0$. It is upon

---

* *Rendiconti del Circolo Matematico di Palermo*, vol. 2 (1888), p. 197, or see Borel's *Théorie des fonctions*, p. 53.

† The conclusions of Poincaré and Borel are not actually inconsistent, but a new point of view is taken by the latter.

‡ *Compt. Rend.*, vol. 128 (1899), p. 78.

this interesting problem that our attention will be especially focused in the first two lectures. In applying henceforth the term divergent to power series, I shall restrict it to series having a zero-radius of convergence.

I shall offer no excuse for any irregularity or incompleteness of treatment. The admirable treatise by *Borel* on *Les Séries divergentes* (1901) and the masterly little book of *Hadamard, La Série de Taylor et son prolongement analytique* (1901), leave little or nothing to be desired in the line of systematic development. While it is impossible not to repeat much that is found in these books, I have also supplemented with other material and sought to give as fresh a presentation as possible.

Lecture 1. *Asymptotic Convergence.*

Few more notable instances of the difference between theoretical and practical mathematics are to be found than in the treatment of divergent series. After the dawn of exact mathematics with *Cauchy* the theoretical mathematician shrank with horror from the divergent series and rejected it as a treacherous and dangerous tool. The astronomer, on the other hand, by the exigencies of his science was forced to employ it for the purpose of computation. The very notion of convergence is said by *Poincaré** to present itself to the astronomer and to the mathematician in complementary or even contradictory aspects. The astronomer requires a series which converges rapidly at the outset. He cares not what the ultimate character may be, if only the first few terms, twenty for example, suffice to compute the desired function to the degree of accuracy required. Consequently he judges the series by these terms. If they increase, the series is for him non-convergent. To the mathematician the question is not at all concerning the nature of the series *ab initio,* but solely concerning its ultimate character.

Let me illustrate the difference by referring to *Bessel's* series

$$J_n = \frac{x^n}{2^n n!}\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)\,(2n+4)} - \cdots\right),$$

* *méthodes nouvelles de la mécanique céleste,* vol. 2, p. 1.

which is a solution of the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0. \tag{2}$$

This is convergent for all values of x, but when x is very large the series is worthless for computation owing to the rapid and long-continued increase of the terms before the convergence finally sets in. The astronomer and physicist therefore have been driven to use for large values of x an expansion which is of the form *

$$Ax^{-\frac{1}{2}}\sin x\left(A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\cdots\right) + Bx^{-\frac{1}{2}}\cos x\left(B_0+\frac{B_1}{x}+\frac{B_2}{x^2}+\cdots\right),$$

or, what is the same thing,

$$Ce^{ix}x^{-\frac{1}{2}}\left(C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\cdots\right) + De^{-ix}x^{-\frac{1}{2}}\left(D_0+\frac{D_1}{x}+\frac{D_2}{x^2}+\cdots\right). \tag{3}$$

Here the multipliers of C and D are only formal solutions of the differential equation (2). In respect to convergence they have a character exactly opposite to that of J_n, since for very large values of x the terms at first decrease rapidly but finally an increase begins. At this point the computer stops and obtains a good approximate value of J_n.

What is the significance of this? It is strange indeed that no attempt was made to study the question until 1886, when *Poincaré* † and *Stieltjes* ‡ simultaneously took it up. That so evident and important a problem should have been so long ignored by the mathematician emphasizes strongly the need of closer touch between him and the astronomer and the physicist. Both *Poincaré* and *Stieltjes* regarded the series as the asymptotic representation

* See, for example, Gray and Mathew's *Treatise on Bessel Functions*, chap. 4.

† *Acta Math.*, vol. 8, p. 295 ff.

‡ *Thesis, Ann. de l'Ec. Nor.*, ser. 3, vol. 3, p. 201.

of one or more functions. While the latter writer studied carefully certain divergent series of special importance with the object of obtaining from the series a yet closer approximation to the function by a species of interpolation, *Poincaré* developed the idea of asymptotic representation into a general theory.

To explain this theory * and at the same time to develop certain aspects scarcely considered by *Poincaré,* I shall start with the genesis of a Taylor's series. Take an interval $(0, a)$ of the positive real axis, and denote by $f(x)$ any real function which is continuous and has $n+1$ successive derivatives at every point within the interval. No hypothesis need be made concerning the character of the function at the extremities of the interval except to suppose that $f(x), f'(x), \cdots, f^{(n)}(x)/n!$ have limiting values $a_0, a_1, \cdots, a_n$ when x approaches the origin. Thus the function at any point within the interval will be represented by Taylor's formula :

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x)$$
$$(0 < \theta < 1).$$

If the function is unlimitedly differentiable and limiting values of $f^{(n)}(x)/n!$ exist for all values of n when x approaches 0, the number of terms in the formula can be increased to any assigned value. Thus the function gives rise formally to a series

$$(1) \qquad a_0 + a_1 x + a_2 x^2 + \cdots,$$

uniquely determined by the limiting values of the function and its derivatives.

The converse conclusion, that the series determines uniquely a function fulfilling the conditions above imposed in some small interval ending in the origin, can not, however, be drawn. This is not even the case when the series is convergent. Suppose, for example, that $a_n = 0$ for all values of n. Then in addition to

* Cf. Peano, *Atti della R. Accad. delle Scienze di Torino,* vol. 27 (1891), p. 40 ; reproduced as *Anhang III* ("Ueber die Taylor'sche Formel") in Genocchi-Peano's *Differential- und Integral-Rechnung,* p. 359.

$f(x) \equiv 0$ we have the functions $e^{-1/x}$, e^{-1/x^2}, $\cdots$, which fulfill the assigned conditions. They are, namely, unlimitedly differentiable within a positive interval terminating in the origin, and when x approaches the origin from within this interval, the functions and their derivatives have the limit 0. From this it follows immediately that if values other than zero be prescribed for the a_n, the function will not be uniquely determined, since to any one determination we may add constant multiples of $e^{-1/x}$, e^{-1/x^2}, $\cdots$.

Inasmuch as the correspondence between the function and the series is not reversibly unique, the series can not be used, in general, for the computation of the value of the generating function. Nevertheless, although this is the case, the series is not without its value. For consider the first m terms, m being a fixed integer. If x is sufficiently diminished in value, each of these terms can be made as small as we choose in comparison with the one which precedes it, and the series therefore at the beginning has the appearance of being rapidly convergent, even though it be really divergent. Evidently also as x is decreased, it has this appearance for a greater and greater number of terms, if not throughout its entire extent. Now by hypothesis the generating function was unlimitedly differentiable within the interval, and the successive derivatives are consequently continuous within $(0, a)$. Hence if the interval is sufficiently contracted, $f^{(m+1)}(x)/(m+1)!$ can be made as nearly equal to a_{m+1} throughout the interval as is desired. We have then for the remainder in Taylor's formula:

$$(4) \qquad R_{m+1}(x) = \frac{f^{(m+1)}(\theta x)}{(m+1)!} x^{m+1} = a_{m+1} x^{m+1}(1+\zeta) \qquad (|\zeta| < \epsilon),$$

in which ϵ is an arbitrarily small positive quantity. Consequently if the first $m+1$ terms of the series should be used to compute the value of the generating function, the error committed would be approximately equal to the next term, provided x be taken sufficiently small.

In these considerations there is, of course, nothing to indicate when x is sufficiently small for the purpose. If the result holds

? when is x "sufficiently small,

simultaneously for a large number of consecutive values of m, the best possible value for the function consistent with our information would evidently be obtained by carrying the computation until the term of least absolute value is reached and then stopping. Herein is probably the justification for the practice of the computer in so doing.

Equation (4) which gave a *limit to the error in stopping with the* $(m+1)$*th term* shows also that *this limit grows smaller as* x *diminishes.* Since, furthermore, by increasing m sufficiently the $(m+2)$th term of (1) may be made small in comparison with the $(m+1)$th term, it is clear that on the whole, *as* x *diminishes, we must take a greater and greater number of terms to secure the best approximation to the function.* These two facts may be comprised into a single statement by saying that the approximation given by the series is of an asymptotic character. This will hold whether the series is convergent or divergent.

This notion can be at once embodied in an equation. From (4) we have

$$(5) \qquad \lim_{x=0+} \frac{f(x) - a_0 - a_1 x - \cdots - a_m x^m}{x^m}$$

$$= \lim_{x=0+} \frac{R_{m+1}(x)}{x^m} = 0 \qquad (m = 1, 2, \cdots).$$

This equation is an exact equivalent of the two properties just mentioned and is adopted by *Poincaré** as the definition of asymptotic convergence. More explicitly stated, the series (1) is said by him to represent a function $f(x)$ asymptotically when equation (5) holds for all values of m.

It will be noticed that this definition omits altogether the assumptions concerning the nature of the function with which we started in deriving the series. Not only has the requirement of unlimited differentiability within an interval been omitted but the existence of right-hand limits for the derivatives as x approaches the origin is not even postulated. If the value a_0 be assigned to

* Loc. cit.

the function at the origin, it will have a first derivative, a_1, at this point but it need not have derivatives of higher order.*

The exclusion of the requirement of differentiability has undoubtedly its advantages. It enlarges the class of functions which can be represented asymptotically by the same series. It also simplifies the application of the theory of asymptotic representation, and this is perhaps the chief gain. The results of *Poincaré's* theory can readily be surmised. The sum and product of two functions represented asymptotically by two given series are represented asymptotically by the sum- and product-series respectively, and the quotient of the two functions will be represented correspondingly, provided the constant term of the divisor is not 0. Also if $f(x)$ is any function represented by the series (1), whether convergent or divergent, and

$$\phi(x) = b_0 + b_1x + b_2x^2 + \cdots$$

is a second series having a radius of convergence greater than $|a_0|$, the asymptotic representation of $\phi[f(x)]$ will be the series which is obtained from

$$b_0 + b_1(a_0 + a_1x + \cdots) + b_2(a_0 + a_1x + \cdots)^2 + \cdots$$

by rearranging the terms in ascending powers of x. Lastly, the integral of $f(x)$ will have for its asymptotic representation the term by term integral of (1). But the correspondence of the function and series may be lost in differentiation, for even if the function permits of differentiation, its derivative will not necessarily be a function having an asymptotic power series. Examples of this kind can be readily given.†

* The ordinary definition of an nth derivative is here assumed. If, however, we define the second derivative by the expression

$$f''(0) = \lim_{x=0} \frac{f(2x) - 2f(x) + f(0)}{x^2}$$

and the higher derivatives in similar fashion, the function must have derivatives of all orders.

† Cf. Borel, *Les Séries divergentes*, p. 35.

This failure is on many accounts an unfortunate one. If a further development of *Poincaré's* theory is to be made — and this seems to me both a possibility and a desirability — his definition probably should be restricted by requiring (a) that the function corresponding to the series shall be unlimitedly differentiable in some interval terminating in the origin, and (b) that the derivatives of the function should correspond asymptotically to the derivatives of the power series. These demands are satisfied in the case of an analytic function defined by a convergent series and seem to be indispensable for an adequate theory of divergent series.*

Thus far we have considered asymptotic representation only for a single mode of approach to the origin. Suppose now that an analytic function of a complex variable x is represented by (1) for all modes of approach to the origin, and let a_0 be the value assigned to the function at this point. Then if the function is one-valued and analytic about the origin, it must also be analytic at this point since it remains finite. Hence the series must be convergent.

The case which has an interest therefore is that in which the asymptotic representation is limited to a sector terminating in the origin. Suppose then that (1) is a given divergent series, and let a function be sought which fulfills the following conditions: (a) the function shall be analytic within the given sector for values of

---

* These requirements are formulated from a mathematical standpoint with a view to extending the theory of analytic functions, and doubtless will be too stringent for various astronomical investigations. Prof. E. W. Brown suggests that for such investigations the conditions might perhaps be advantageously modified by making the requirements for only m derivatives, m being a number which varies with x and increases indefinitely upon approach to the critical point. He also points out the difficulties of an extension in the case of numerous astronomical series which have the form $f(x, t) = a_0 + a_1 x + a_2 x^2 + \cdots$, where a_i is a function of x and t, $\partial f / \partial t$ being a convergent series. Poincaré's definition is however still applicable.

Oftentimes in celestial mechanics the only information concerning the function sought is afforded in the approximation given by the asymptotic series. An objection to Poincaré's definition is that it presupposes a knowledge of the function sought, for example, that $\lim f(x) = a_0$, when $x = 0$. As a matter of fact the properties are often unknown. See in this connection p. 89 of these lectures.

x which are sufficiently near to the origin; (b) it shall be represented asymptotically by the given series within the sector, whether inclusive or exclusive of the boundary will remain to be determined; (c) the asymptotic representation shall not be valid if the angle of the sector is enlarged. So far as I am aware, the existence of a function or of functions which meet these requirements has never been demonstrated, though it seems likely that they in general exist. It is, however, very possible that the sector must be restricted in position as well as in magnitude. It may be found necessary to require that the interior of the sector shall not include certain arguments of x; for example, in the case of the series $\Sigma m!x^m$ * the argument 0, for which the terms have all the same sign. † If this be true, the sector will very probably have two such arguments for its boundaries. When there is a function which satisfies the conditions imposed, it can not be unique. For clearly $e^{-1/x}$, $e^{-1/x^{\frac{1}{2}}}$, $e^{-1/x^{\frac{1}{3}}}$, $\cdots$, within certain sectors of angle π, 2π, 3π, $\cdots$, have an asymptotic series in which each coefficient is 0. If, then, any function has been obtained satisfying the conditions stated, one or more of these exponentials, after multiplication by suitable constants, may be added to the function without destroying its properties. Hence if a divergent series is to represent a function uniquely, supplementary conditions must be imposed. The nature of these conditions has not yet been ascertained. ‡

In closing the general discussion a simple extension of the notion of asymptotic convergence should be mentioned which is necessary for the applications to follow. $F(x)$ is said to be represented asymptotically by

$$\Phi(x)\{a_0 + a_1x + a_2x^2 + \cdots\}$$

---

* This series is discussed in the next lecture.

† Borel (*loc. cit.*, p. 36) in his exposition of Poincaré's theory seems to make the definite statement that there are arguments for which no corresponding function exists, but I am unable to find any proof of the statement.

‡ In this connection see pp. 89–92 of Borel's article, *Ann. de l'Ec. Nor.*, ser. 3, vol. 16 (1899).

when the series in parenthesis gives such a representation of $F(x)/\Phi(x)$.

The applications of *Poincaré's* theory have been made chiefly in the province of differential equations * where divergent series are of very common occurrence. We will take for examination the class of equations, of which the theory is perhaps the most widely known, the homogeneous linear differential equation with polynomial coefficients:

$$(6) \qquad P_n(x)\frac{d^ny}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_0(x)y = 0.$$

This is, in fact, the class of equations to which *Poincaré* first applied his theory,† but his discussion of the asymptotic representation of the integrals was limited to a single rectilinear mode of approach to the singular point under consideration. The determination of the sectors of validity for the asymptotic series has been made by *Horn*,‡ who in a number of memoirs has carefully studied the application of the theory to ordinary differential equations.§

As is well known, the only singular points of (6) are the roots of $P_n(x)$ and the point $x = \infty$. For a regular singular point ‖ we have the familiar convergent expressions for the integrals given by Fuchs. Consider now an irregular singular point. By a linear transformation this point may be thrown to ∞, the equation being still kept in the form (6). Suppose then that this has been done. If P_n is of the pth degree, the condition that $x = \infty$ shall be a regular singular point is that the degrees of $P_{n-1}, P_{n-2}, \cdots, P_0$ shall be at most equal to $p-1, p-2, \cdots, p-n$, respectively.

For an irregular singular point some one or more of the degrees must be greater. Let h be the smallest positive integer for which the degrees will not exceed successively

* In addition to the memoirs cited below Poincaré's *Les méthodes nouvelles de la mécanique céleste* and various memoirs by Kneser may be consulted.

† *Acta Math.*, vol. 8 (1886), p. 303. See also *Amer. Jour.*, vol. 7 (1885), p. 203.

‡ *Math. Ann.*, vol. 50 (1898), p. 525.

§ See various articles in *Crelle's Journal* and the *Mathematische Annalen*.

‖ *Stelle der Bestimmtheit.*

$$p + (h-1), \quad p + 2(h-1), \quad p + 3(h-1), \cdots.$$

The number h is called the rank of the singular point ∞, and the differential equation can be satisfied formally by the series of *Thomae* or the so-called normal series:

$$(7) \qquad S_i = e^{\frac{\alpha_i x^h}{h} + \alpha_{i,1} x^{h-1} + \cdots + \alpha_{i,h-1} x} x^{\rho_i} \left(C_i + \frac{C_{i,1}}{x} + \frac{C_{i,2}}{x^2} + \cdots \right)$$
$$(i = 1, 2, \cdots, n).$$

Unless certain exceptional conditions are fulfilled, there are n of these expansions, and in general they are divergent. To simplify the presentation let us confine ourselves to the case for which $h = 1$. Then at least one of the polynomials succeeding P_n will be of the pth degree, and none of higher degree. Place

$$P_n = A_n x^p + B_n x^{p-1} + \cdots,$$
$$P_{n-1} = A_{n-1} x^p + B_{n-1} x^{p-1} + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$P_0 = A_0 x^p + B_0 x^{p-1} \cdots,$$

and construct the equation

$$(8) \qquad A_n \alpha^n + A_{n-1} \alpha^{n-1} + \cdots + A_0 = 0.$$

The n roots of this equation are the n quantities α_i which appear in the exponential components of the S_i.

As a particular illustration of the class of equations under consideration, *Bessel's* equation (Eq. (2)) may be cited. Here the point ∞ is of rank 1, the characteristic equation is

$$A_0 \alpha^2 + A_1 \alpha^2 + A_2 \equiv \alpha^2 + 1 = 0,$$

with the roots

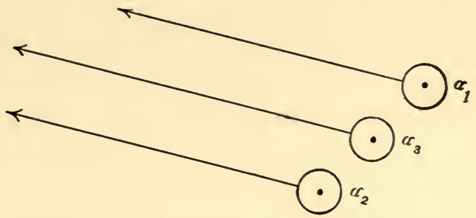
$$\alpha_1 = -i, \quad \alpha_2 = +i,$$

and the two *Thomaean* integrals are

$$(9) \qquad \begin{aligned} y_1 &= e^{ix} x^{\rho_1} \left(C_0 + \frac{C_1}{x} + \cdots \right), \\ y_2 &= e^{-ix} x^{\rho_2} \left(D_0 + \frac{D_1}{x} + \cdots \right), \end{aligned}$$

in which ρ_1, ρ_2 are yet to be ascertained. After this has been done, the coefficients of (9) can be determined by direct substitution in (2).

To avoid complications we will assume that the n roots of the characteristic equation (8) are all distinct, also that the real parts of no two roots are equal. Mark now in the complex plane the points $\alpha_1, \alpha_2, \cdots, \alpha_n$, and draw from them to infinity a series of parallel rays having such a direction that no one of the rays with its prolongation in the opposite direction shall contain two or more of these points. Finally surround the points α_i with small circles,



so that we shall have the familiar loop circuits for the paths of integration of the integrals which we now proceed to form. Put

$$\eta_i = \int e^{zx} v_i(z) dz \qquad (i = 1, \cdots, n), \tag{10}$$

in which $v_i(z)$ is a function to be subsequently fixed. In order that the integral may have a sense, x will be so restricted that the real part of zx shall be negative for the rectilinear parts of the loop circuits. We can then so determine $v_i(z)$ that η_i shall be a solution of (6).

For this purpose substitute η_i for y in (6). A reduction, based on the integration of (10) by parts,* gives for $v_i(z)$ the equation

$$(A_n z^n + A_{n-1} z^{n-1} + \cdots + A_0) \frac{d^p v}{dz^p} + \cdots + (\ \ ) v = 0, \tag{11}$$

This is known as *Laplace's* transformed equation. While the original equation was of the nth order with coefficients of the pth

* Cf. Picard's *Traité d'Analyse*, vol. 3, p. 383 ff., or Poincaré, *Amer. Jour.*, vol. 7 (1885), p. 217 ff.

degree, the transform is of the pth order with coefficients of the nth degree. Its singular points in the finite plane are the roots of the first coefficient of (11), which is identical with the left hand member of (8). Furthermore, an inspection of (11) shows immediately that each of these singular points α_i is regular, and the exponents which belong to it are

$$0,\ 1,\ 2,\ \cdots,\ p-2,\ \beta_i \equiv -(\rho_i+1) \qquad (i=1,\ 2,\ \cdots,\ n),$$

in which ρ_i is the exponent of x, hitherto undetermined in (7). Hence if β_i is not an integer, there is an integral of (11) having the form

$$(z-\alpha_i)^{\beta_i}(k_0+k_1(z-\alpha_i)+k_2(z-\alpha_i)^2+\cdots),$$

which, when continued analytically, can be taken as the function v_i. Thus for the solution of (6) we obtain

$$\eta_i=\int e^{zx}(z-\alpha_i)^{\beta_i}(k_0+k_1(z-\alpha_i)+\cdots)dz.$$

If, finally, α_i+y/x is substituted for z the integral becomes

$$(12)\qquad \eta_i=e^{\alpha_i x}x^{-\beta_i-1=\rho_i}\int e^y y^{\beta_i}\left(k_0+k_1\frac{y}{x}+k_2\frac{y^2}{x^2}+\cdots\right)dy,$$

where the transformed path of integration is a loop circuit which encloses the origin of the y-plane, the rectilinear portion of the path lying in the half plane for which the real part of y is negative.

We have thus reached a solution of the differential equation under the form of an improper integral of a convergent series. The integration of (12) term by term, which is a purely formal process, gives at once the normal integral S_i of (7), in which

$$C_{i,\,n}=k_n\int e^y y^{\beta_i+n}dy.$$

The asymptotic character of S_i can be quickly demonstrated.* For let $u^n R_n(u)$ denote the remainder after n terms of the series

$$k_0+k_1u+k_2u^2+\cdots.$$

Then

* Horn, *loc. cit.*, or *Acta Math.*, vol. 24 (1901), pp. 299 ff.

$$x^n\left\{\eta_i e^{-a_i x} x^{-\rho_i} - \left(C_i + \frac{C_{i,1}}{x} + \cdots + \frac{C_{i,n}}{x^n}\right)\right\}$$
$$= \frac{1}{x}\int e^y y^{\beta_i+n+1} R_{n+1}\left(\frac{y}{x}\right) dy.$$

Since the integral in the right hand member, taken along the loop circuit, can be shown to remain finite when $x = \infty$, we have

$$\lim_{x=\infty} x^n\left\{\eta_i e^{-a_i x} x^{-\rho_i} - \left(C_i + \frac{C_{i,1}}{x} + \cdots + \frac{C_{i,n}}{x^n}\right)\right\} = 0.$$

But this is the statement of *Poincaré's* definition of asymptotic convergence for $x = \infty$.

I have sketched this lengthy process in some detail because it is a thoroughly typical one and indicates the present status of the theory of asymptotic series. It will be observed that the follow ing course is pursued:

1. First, it is discovered that the differential equation permits of formal solution by a certain divergent series.

2. By some independent process the existence of an actual solution is ascertained which permits formally of expansion into the series. Usually the solution is found under the form of an integral, and *Horn* has applied the theory chiefly in cases in which solutions of this form were known. (Lately, however, he has used solutions obtained from the differential equation by the process of successive approximation.*)

3. The asymptotic character of the series is then argued and, finally, the sector within which this representation is valid is determined.

The status of the theory thus exhibited seems to me an unsatisfactory and transitional one. It is to be hoped that ultimately the theory will be so developed that the mere existence of a divergent power series as a *formal* solution of the differential equation will be sufficient for the immediate affirmation of the existence of one or more solutions which are analytic functions with certain specified properties.

* *Math. Ann.*, vol. 51 (1898), p. 346. In *Crelle's Journal*, vol. 118 (1897), still another method is used for obtaining the solutions.

It remains yet to fix the sectors within which the solutions η_i can be represented asymptotically by the normal integrals. These sectors have been specified by *Horn** in the following manner. Let straight lines be drawn from each singular point a_i to every other point and produce each joining line to infinity in both directions. A set of lines will be thus fixed, radiating from the point ∞. Let their arguments, taken in the order of decreasing magnitude, be denoted by

$$\omega_1, \omega_2, \cdots, \omega_r, \omega_{r+1} = \omega_1 - \pi, \cdots, \omega_{2r} = \omega_r - \pi.$$

Suppose now that the argument of the rectilinear part of the path of integration for η_i in the plane of z lies between $\omega_{\rho-1}$ and ω_ρ. Then η_i is represented asymptotically by S_i for values of the argument of x between $\pi/2 - \omega_{\rho-1}$ and $\pi/2 - \omega_{\rho+r}$.†

To the general solution of (6), $c_1\eta_1 + c_2\eta_2 + \cdots + c_n\eta_n$, there corresponds the divergent expansion

$$\begin{aligned} c_1S_1 + \cdots + c_nS_n = c_1e^{a_1x}x^{\rho_1}\left(C_1 + \frac{C_{1,1}}{x} + \frac{C_{1,2}}{x^2} + \cdots\right) + \\ \cdots + c_ne^{a_nx}x^{\rho_n}\left(C_n + \frac{C_{n,1}}{x} + \frac{C_{n,2}}{x^2} + \cdots\right). \end{aligned} \tag{13}$$

Here the real parts of two exponents, a_ix and a_jx, are equal only when $arg\,(a_i - a_j)x$ is an odd multiple of $\pi/2$; that is, when $arg\,x$ is equal to $\pi/2 - \omega_i\,(i = 1, \cdots, 2r)$. Suppose then that for

$$\pi/2 - \omega_{\rho-1} < arg\,x < \pi/2 - \omega_{\rho+r}$$

we so assign subscripts to the a_i that

$$R(a_1x) > R(a_2x) > \cdots > R(a_nx).$$

Then all the integrals for which $c_1 \neq 0$ have in common the asymptotic series c_1S_1, while those for which $c_1 = c_2 = \cdots = c_{i-1}$,

* Horn, *Math. Ann.*, vol. 50 (1898), p. 531.

† In certain cases the asymptotic representation may be valid for a greater range of values of the argument of x, as in the case of Bessel's equation discussed below.

$c_i \neq 0$, are represented by $c_i S_i$. Thus it appears that between the arguments considered S_n is the only one of the n asymptotic series S_i which defines a solution of the differential equation (6) uniquely.

Changes in the asymptotic series representing a solution may occur from two causes, either because x passes through one of the critical values above mentioned for which there is a change in the dominant exponential in (13), or because of a sudden alteration in the values of the constants c_i for certain values of the argument. This can be made clear, in conclusion, by illustrating with *Bessel's* equation.* For this equation, as we saw,

$$\alpha_1 = -i, \quad \alpha_2 = +i,$$

and hence

$$\omega_1 = \frac{3\pi}{2}, \quad \omega_2 = \frac{\pi}{2}.$$

Also since *Laplace's* transform for the particular case before us is †

$$(z^2 + 1)\frac{d^2v}{dz^2} + 3z\frac{dv}{dz} + (\quad)v = 0,$$

the exponent ρ_i for either of the two singular points $z = \pm i$ has the value $-\frac{1}{2}$. Accordingly the series (13) for $c_1\eta_1 + c_2\eta_2$ may be written

$$Ce^{ix}x^{-\frac{1}{2}}\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots\right)$$
$$+ De^{-ix}x^{-\frac{1}{2}}\left(D_0 + \frac{D_1}{x} + \frac{D_2}{x^2} + \cdots\right) = CU(x) + DV(x),$$

as previously given in (3). If the imaginary part of x is negative, $CU(x)$ is the dominant term in (3) and gives the asymptotic representation of the general solution, $c_1\eta_1 + c_2\eta_2$. On the other hand, if the imaginary part is positive, the dominant term is

* A brief but very interesting discussion is given in a letter of Stokes in the *Acta Math.*, vol. 26 (1902), pp. 393–397. Compare also § 3 of Horn's article, *Math. Ann.*, vol. 50 (1898), p. 525.

† *Math. Ann.*, vol. 50, p. 539, Eq. B'.

$DV(x)$. The changes in the values of C and D take place only when $\arg x$ passes through the values $(2n+1)\pi/2$. Then the coefficient of the dominant term remains unaltered, while the coefficient of the inferior term is altered by an amount proportional to the coefficient of the dominant term.† We conclude, therefore, that in general the asymptotic series for any solution of *Bessel's* equation changes abruptly for values of the argument congruent with $0 \pmod{\pi}$. Furthermore, the series can not be valid for a greater range of values of the argument unless when $\arg x = 0$, either $D = 0$ or $C = 0$. In the former case we have a particular solution $C\eta_1$ which is represented by the series $CU(x)$ for

$$-\pi < \arg x < 2\pi,$$

and in the latter case a solution $D\eta_2$ represented by $DV(x)$ for

$$-2\pi < \arg x < \pi.$$

Thus from the infinitely many solutions of *Bessel's* equation having the common asymptotic representation $CU(x)$ and $DV(x)$ respectively, these two solutions can be singled out by the requirement that the asymptotic representation shall have the maximum sector of validity.

Lecture 2. *The Application of Integrals to Divergent Series.*

In the first lecture a divergent series was connected with a group of functions, for which it afforded a common asymptotic representation. In the present lecture I shall treat of methods which have been used to derive a function uniquely from the series. To establish, whenever possible, such a unique connection, to develop the properties of the function, and to determine the laws and conditions under which the series can be manipulated as a substitute for the function — this may be said to be the ultimate aim of the theory of divergent series.

Up to the present time this goal has been reached only for a restricted class of divergent series. Furthermore, the uniqueness

† Stokes, *loc. cit.*

of correspondence between the function and the series has been attained, not by a specification of the properties of the function, but by means of some algorithm which, when applied to the series, yields a single function. Unquestionably the instrument by which the greatest progress has been made thus far is the integral. The first successes, however, were reached by *Laguerre* * and *Stieltjes* † through the use of continued fractions, and very possibly in the end the continued fraction will prove to be the best, as it was the earliest tool. But as yet it has been applied only in cases in which the function can be represented under the form of an integral as well as of a continued fraction, although with greater difficulty.

To explain the use of integrals let us consider the familiar divergent series treated by *Laguerre*, Vol. 1.

$$(1) \qquad 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots.$$

This is, I believe, historically the first divergent series from which a functional equivalent was derived. ‡ Since

---

* See No. 20 of the bibliography at the end of lecture 6.

† Bibliography, No. 26a.

‡ Laguerre (*loc. cit.*) gives the function first in the form of a continued fraction and later proves its identity with the integral which gives rise to the divergent series. Borel at the opening of the second chapter of *Les Series divergentes* remarks that "Laguerre parait avoir le premier montré nettement l'utilité qu'il peut y avoir à transformer une série divergente . . . en une fraction continue convergente." It seems almost to have escaped notice (see, however, p. 110 of Pringsheim's report, *Encyklopädie der Math. Wissenschaften*, I A 3), that Euler (Bibliography, No. 46) derived a continued fraction from the divergent series

$$1 + mx + m(m+n)x^2 + m(m+n)(m+2n)x^3 + \cdots,$$

of which Laguerre's series is a special case, and clearly realizes the utility of the continued fraction. Moreover, a close parallel to the course followed by Laguerre is found in the work of Laplace who derives from the expression

$$e^{x^2}\int_x^\infty e^{-x^2}dx$$

a divergent series and from this in turn a continued fraction, the convergents of which were stated by him and proved by Jacobi to be alternately greater and less than the expression. Had Jacobi proved also the convergence of the continued fraction, the work of Laguerre would have had an exact parallel for real values of x. Cf. No. 47 of the bibliography.

$$m! = \Gamma(m+1) = \int_0^\infty e^{-z} z^m dz,$$

the series may be written

$$\int_0^\infty e^{-z} dz + x\int_0^\infty e^{-z} z dz + x^2 \int_0^\infty e^{-z} z^2 dz + \cdots,$$

the path of integration being the positive real axis. If, then, by a merely formal process, the sum of the integrals is replaced by the integral of the sum, we obtain

$$\int_0^\infty e^{-z}(1 + xz + x^2z^2 + \cdots)\, dz,$$

or a function

(2) $$f(x) \equiv \int_0^\infty e^{-z} F(zx) dz,$$

in which

$$F(zx) = \frac{1}{1 - zx}.$$

The function thus derived is an improper integral which has a significance for all values of x except those which are real and positive. It can be shown also to be analytic for all except the excluded values of x. One of the simplest proofs is as a corollary of the following exceedingly fundamental theorem of *Vallée-Poussin*,* which we shall have occasion to use again later: *If in the proper integral*

$$\int_a^b f(x, z) dz$$

the integrand is continuous in z and x for all values of z upon the path of integration and for all values of x within a region T; if, furthermore, for each of the above values of z it is analytic in x over the region T, the integral will also be an analytic function of x in the interior of T. By this theorem, if t is a point on the positive real axis,

$$\int_0^t \frac{e^{-z} dz}{1 - zx}$$

* *Ann. de la Soc. Scient. de Bruxelles*, vol. 17 (1892–3), p. 323.

will represent an analytic function of x over any closed region of the x-plane which excludes the positive real axis. If, now, t passes through any indefinitely increasing set of values,

$$t_1 < t_2, < t_3, \cdots,$$

we have in

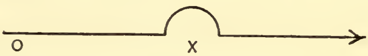
$$f_i(x) = \int_0^{t_i} \frac{e^{-z}dz}{1 - zx}$$

a series of analytic functions which is seen at once to converge uniformly over the region considered, since

$$|f_i(x) - f_j(x)| \equiv \left| \int_{t_j}^{t_i} \frac{e^{-z}dz}{1 - zx} \right| < \epsilon$$

for sufficiently great values of i and j. The limit (2) is therefore analytic.

By deforming the path of integration the same conclusion concerning the analytic character of the function (2) can be extended to all values of x upon the positive real axis excepting 0 and ∞, and when the deformation is made on opposite sides of a fixed point x, the two values of the integral will be found to differ by



$$(3) \qquad 2i\pi \frac{1}{x} e^{-\frac{1}{x}}.$$

The integral accordingly represents a multiple-valued function with the singular points 0 and ∞, the various branches of which differ from one another by multiples of the period (3). For the initial branch which was given in (2) the limit of $f^{(n)}(x)/n!$ will be the $(n + 1)$th coefficient of (1) if x approaches the origin along any rectilinear path except the positive real axis.

Let the process which has been adopted for the series of *Laguerre* be applied next to any other series

$$(\mathrm{I}) \qquad a_0 + a_1x + a_2x^2 + \cdots$$

having a finite radius of convergence. If we write the series in the form

$$a_0 + a_1 x + 2\,!\left(\frac{a_2}{2\,!}\right)x^2 + \cdots + n\,!\left(\frac{a_n}{n\,!}\right)x^n + \cdots,$$

then replace the factor $n\,!$ by its expression as a Γ-integral, and finally, by a step having in general only formal significance, bring all the terms under a common integral sign, we shall obtain

$$\int_0^\infty e^{-z}\left(1 + a_1 xz + \frac{a_2}{2\,!}x^2z^2 + \cdots\right)dz$$

or

(4) $$\int_0^\infty e^{-z}F(zx)dz,$$

in which

(5) $$F(u) = 1 + a_1 u + \frac{a_2}{2\,!}u^2 + \cdots + \frac{a_n}{n\,!}u^n + \cdots \qquad (u = zx).$$

This integral is the expression upon which *Borel* builds his theory of divergent series, and may be regarded as a generalization of a very interesting theorem of *Caesaro.** The series (5) is called the *associated series* of (I).

Two cases are now to be distinguished according as the fundamental series (I) has, or has not, a radius of convergence R which is greater than 0. If the radius is not zero, the associated series has an infinite radius since

$$\lim_{n=\infty}\sqrt[n]{\frac{a_n}{n\,!}} = \lim_{n=\infty}\sqrt[n]{\frac{R^{-n}(1+\epsilon)^n}{n\,!}} = 0,$$

and it accordingly represents an entire function. It is a simple matter to prove that the integral (4) will have a sense if x lies within the circle of convergence of (I), and that the values of the integral and series are identical. But the integral may also have a sense for values of x which lie without the circle, and in this case the integral may be used to get the analytic continuation of (I).

* Cf. Borel, *Les Séries divergentes*, pp. 88–98.

The series is said by *Borel* to be *summable* * at a point x when the integral (4) has a meaning at this point.

The second case is that in which the fundamental series is divergent. The associated series in this case may be either convergent or divergent. If it is convergent only over a portion of the plane of $u = zx$, we are to understand by $F(u)$ not merely the value of the associated series but of its analytic continuation. Let x for an instant be given a fixed value. Then when z describes the positive real axis, u in its plane describes the ray from the origin passing through the point x. If $F(u)$ is holomorphic along this ray, it is possible that the integral (4) will have a sense. Suppose that this holds good as long as x lies within a certain specified region of its plane. Then for this region a function will be obtained uniquely from the divergent series by the use of the integral, precisely as in the case of the series of *Laguerre*.

This method of treatment is obviously restricted to divergent series for which the associated series are convergent, and it will not always be applicable even to these. A divergent series in which there is an infinite number of coefficients of the same order of magnitude as the corresponding coefficients of

$$(6) \qquad 1 + x + (2\,!)^2x^2 + (3\,!)^2x^3 + \cdots + (n!)^2x^n + \cdots$$

can not be summed in this manner. It will be noticed, however, that the series just given is one whose first associated series is the series of *Laguerre*, and whose second associated series is consequently convergent.

The method of *Borel* can be readily extended so as to take account also of such series, or, more generally, of series that have an associated series of the nth order which is convergent. One mode of doing this is by the introduction of an n-fold integral. Suppose, for example, that in (6) one of the two factorials $n\,!$ is replaced by

$$\int_0^\infty e^{-z}z^n dz$$

* Some other term would be preferable since his definition refers only to one of many possible modes of summation. A series may be simultaneously "summable" at a point x by one method, and non-summable by another.

and the other by

$$\int_0^\infty e^{-t}t^n dt.$$

The $(n+1)$th term of the series becomes

$$x^n \int_0^\infty \int_0^\infty e^{-t-z} z^n t^n dz dt,$$

and we obtain the two-fold integral

$$\int_0^\infty \int_0^\infty \frac{e^{-t-z}}{1-tzx} dz dt$$

for the functional equivalent of the series. This is a function, the initial branch of which is analytic over the entire plane of x except at the points 0 and ∞.

We turn now to the consideration of the region of summability, in which x must lie in order that the integral shall have a sense. *Borel* has determined the shape of this region when the fundamental series (I) is convergent, but in so doing he restricts himself to what he calls the *absolutely summable* series. The series (I) is said to be *absolutely summable* for any value of x when the integral (4) is absolutely convergent and when, furthermore, the successive integrals

(7) $$\int_0^\infty e^{-z} \left| \frac{d^\lambda F(zx)}{dz^\lambda} \right| dz \qquad (\lambda = 1, 2, \cdots)$$

have also a sense.*

To fix the shape of the region *Borel* shows first that if a function defined by a convergent series (I) is absolutely summable at a point P, it is analytic within the circle described upon the line OP as diameter, connecting P with the origin O; conversely, if it is analytic within and upon a circle having OP as diameter, it must be absolutely summable along OP, inclusive of the point

* The condition (7) was not originally included in Borel's definition of absolute summability (*Ann. de l'Ec. Nor.*, ser. 3, vol. 16, 1899), and is superfluous in fixing the shape of the region. Cf. *Math. Ann.*, vol. 55 (1902), p. 74. The modification of the definition was introduced in the *Séries divergentes* and is needed for the developments explained below, p. 102. Chapters 3 and 4 of this treatise can be read in connection with the present lecture.

P. As P moves outward from the origin along any ray, the limiting position for the circle is one in which it first passes through a singular point S, and at this point SP and OS subtend a right angle. The region of absolute summability can therefore be obtained as follows: Mark on each ray from the origin the nearest singular point of the function defined by (I), if there is such a point in the finite plane. Then through this point draw a perpendicular to the line. Some or all of these perpendiculars will bound a polygon, the interior of which contains the origin and is not penetrated by any one of the perpendiculars. This region is called the *polygon of summability*. If the singularities of the function are a set of isolated points, the polygon will be rectilinear. For the extreme case in which the circle of convergence is a natural boundary, the polygon and circle coincide. In every other case the circle is included in the polygon. Thus by the use of (4) *Borel* effects an analytic continuation of the series over a perfectly definite region whenever an analytic continuation exists. On passing to the exterior of the polygon the series ceases to be absolutely summable. As an example of this result, take the series

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

which is the familiar expansion of $\frac{1}{2} \log (1 + x)/(1 - x)$. The singular points of the function are $+ 1$ and $- 1$, the circle of convergence is the unit circle, and the polygon of summability is a strip of the plane included between two perpendiculars to the real axis through the points ± 1.

When the given series is divergent, the form of the domain of summability has not been determined with such precision. The only information which we have upon the subject is contained in a brief but important communication by *Phragmen* in the *Comptes Rendus*,* published since the appearance of *Borel's* work. *Phragmen* considers here the domain, not of absolute, but of simple summability for *Laplace's* integral

* Vol. 132, p. 1396; June, 1901.

$$(8) \qquad \int_0^\infty e^{-z} f(zx)\, dz,$$

in which $f(zx)$ denotes an arbitrary function.

To adopt a term of *Mittag-Leffler*, the domain is a "star," which is derived as follows: Draw any ray from the origin. If the series is summable at a point x_0 of this line, *Phragmen* shows that it is summable at every point between x_0 and the origin O. There is therefore some point P of the line which separates the interval of summability from the interval of non-summability. If the function is summable for the entire extent of the ray, P lies at infinity. In any case let the segment OP be obliterated and then make a cut along the remainder of the line. When the same thing is done for every ray which terminates at the origin, there is left a region called a star, bounded by a set of lines radiating from a common center, the point at infinity.

Phragmen says that the proof of this result is so simple that it can be given "*en deux mots.*" For this reason I shall reproduce it here. We are to show that if the integral converges for any value $x = x_0$, it will also converge for $x = \theta x_0$, if $0 < \theta < 1$. Place

$$f(zx_0) = \phi(z) + i\psi(z).$$

For $x = x_0$ the real and imaginary components of the integrals,

$$(9) \qquad \int_0^\infty \phi(z) e^{-z} dz, \qquad i\int_0^\infty \psi(z) e^{-z} dz,$$

have a sense. We are to prove that the integrals

$$(10) \qquad \int_0^\infty \phi(z\theta) e^{-z} dz, \qquad \int_0^\infty \psi\,(z\theta) e^{-z} dz,$$

obtained by replacing x_0 by θx_0, also exist. Consider either integral, for example the former. Let $0 < a_1 < a_2 < \infty$, and put

$$J = \int_{a_1}^{a_2} \phi\,(z\theta) e^{-z} dz.$$

By the change of variable $w = \theta z$ this becomes

$$J = \frac{1}{\theta}\int_{\theta a_1}^{\theta a_2} \phi(w) e^{-\frac{w}{\theta}}\, dw = \frac{1}{\theta}\int_{\theta a_1}^{\theta a_2} e^{-w\left(\frac{1}{\theta}-1\right)} \phi(w) e^{-w} dw.$$

Since $e^{-w(1/\theta-1)}$ is a positive and decreasing function in the interval considered, the second mean-value theorem of the integral calculus* may be applied, giving

$$(11) \qquad J = \frac{e^{-a_1(1-\theta)}}{\theta}\int_{\theta a_1}^{\theta a} \phi(w) e^{-w} dw,$$

in which a designates an appropriate value between a_1 and a_2. This, as *Phragmen* says, proves the theorem, but a word or two of explanation additional to his "*deux mots*" may not be unacceptable to some of my hearers. The necessary and sufficient condition for the existence of the first of the two integrals given in (10) is that by taking two values a_1 and a_2 sufficiently small or two values sufficiently large, the integral J may be made as small as we choose. Now this is true of

$$\int_{a_2}^{a} \phi(w) e^{-w} dw$$

since the integrals (9) exist, and equation (11) show then that it must be true likewise of J because the factor $e^{-a_1(1-\theta)}/\theta$ has an upper limit for $0 < \theta_1 < \theta < 1$ and $0 < a_1 < \infty$. It follows therefore that the integrals (10) exist.

Two other facts stated by *Phragmen* are also of interest. The function of x defined by (8) is a monogenic function which is holomorphic at every point in the interior of a circle described upon OP as diameter. If, also, in place of $f(zx)$ we take the associated series $F(zx)$ of a convergent series (I), the star of convergence coincides with *Borel's* polygon of absolute summability. Thus the regions of absolute and non-absolute summability are the same, or differ at most only in respect to the nature of the boundary points.

* Bonnet's form: *Encyklopädie der Math. Wiss.*, II A 2, § 35.

It might be thought that the result of *Phragmen* makes the concept of absolute summability useless. This is, however, in no wise the case. At any rate, *Borel* employs the concept to establish the important conclusion that a divergent series, if absolutely summable, can be manipulated precisely as a convergent series. Thus if two absolutely summable series, whether convergent or divergent, are multiplied together, the resultant series will also be absolutely summable, and the function which it defines will be the sum or product of the functions defined by the two former series. Or, again, if an absolutely summable series is differentiated term by term, another such series is obtained, and the latter yields a function which is the derivative of the one defined by the former series. Lastly, the function determined by an absolutely summable series can not be identically zero, unless all the coefficients of the series vanish.

These facts make possible the immediate application of *Borel's* theory to differential equations. If, in short,

$$P(x, y, y', \cdots, y^{(n)}) = 0$$

is a differential equation which is holomorphic in x at the origin and is algebraic in y and its derivatives, any absolutely summable series (I), which satisfies formally the equation, defines an analytic function that is a solution of the equation. For example, it will be found that the series of *Laguerre* satisfies formally the equation

$$x^2\frac{dy}{dx} + (x - 1)\,y = -1$$

and hence the function

$$f(x) = \int_0^\infty \frac{e^{-z}dz}{1 - zx}$$

must be a solution of the equation.

These conclusions of *Borel* should be strongly emphasized. In any complete theory of divergent series it is an ultimatum that they shall in all essential points * permit of manipulation

* In an absolutely summable series it is not always legitimate to change the order of an infinite number of terms. Cf. Borel, *Journ. de Math.*, ser. 5, vol. 2 (1896), p. 111.

precisely as convergent series, this property being a requisite for satisfactory application to differential equations.

In our preceding exposition of *Borel's* theory, we have introduced his chief integral by a method which permits of expansion in various directions. *Le Roy* in his very excellent thesis * suggests a change of the function in *Laplace's* integral which greatly enlarges the applicability † of *Borel's* method without essentially changing its character. Let the initial series (I) be first written

$$a_0 + a_1\Gamma(p+1)\frac{x}{\Gamma(p+1)} + a_2\Gamma(2p+1)\frac{x^2}{\Gamma(2p+1)} + \cdots$$
$$+ a_n\Gamma(np+1)\frac{x^n}{\Gamma(np+1)} + \cdots,$$

and then replace the second factor in each term by

$$\Gamma(np+1) = \frac{1}{p}\int_0^\infty e^{-z^{1/p}} z^{1/p-1+n}\,dz.$$

This gives for the formal equivalent of the series the integral

$$(12) \qquad \frac{1}{p}\int_0^\infty e^{-z^{1/p}} z^{1/p-1}\,F(zx)\,dz,$$

in which the associated function is now

$$(13) \qquad F(zx) = 1 + \frac{a_1 zx}{\Gamma(p+1)} + \frac{a_2 z^2 x^2}{\Gamma(2p+1)} + \cdots.$$

The number p remains to be fixed. If the series (I) is divergent, there is a critical value of p such that any smaller value of p gives an associated series having a zero radius of convergence, while a larger value gives one with an infinite radius of convergence. This critical value p' may be said to gauge or measure

---

* *Annales de Toulouse*, ser. 2, vol. 2 (1900), p. 416.

† Since this was written, a very interesting application of Le Roy's idea to differential equations has been made by Maillet, *Ann. de l'Ec. Nor.*, ser. 3, vol. 20 (1893), p. 487 ff.

the degree of divergence of the series. For the divergent series treated by *Borel*, $p' \leqq 1$. If $p' = 0$, the series (I) has a finite radius of convergence. On the other hand, when $p' = \infty$, *Le Roy's* integral can not be applied, but it may be conjectured that such cases will be of very rare occurrence. *Le Roy* proposes to employ the integral when the associated series is convergent for $p = p'$ and when also its circle of convergence has a finite radius and is not a natural boundary. The function obtained from (12) will be unique, and he shows that the series which are summable by its use like the series of *Borel*, can be manipulated as convergent series. One might also inquire whether, in case (13) diverges for $p = p'$ and we take $p > p'$, we shall not get a unique result irrespective of the value of p.

Other forms of integrals may also be selected for the summation of the series, as for example,*

$$\int_0^\infty f(z) F(zx) dz,$$

in which

$$F(zx) = \beta_0 + \beta_1 zx + \beta_2 z^2 x^2 + \cdots.$$

To generate the given series (I) we must so select $f(x)$ and $F(zx)$ that

$$a_n = \beta_n \int_0^\infty f(z) \cdot z^n dz.$$

Borel chooses for $f(z)$ the exponential function, making in consequence $F(zx)$, his associated series, dependent only upon the given series. Hence his process is called very appropriately the exponential method of summation. *Stieltjes*,† on the other hand, with his continued fraction arrives at an integral in which $F(u)$ is the fixed function and $f(z)$ is the variable function dependent on the series given. For the fixed function he takes

$$F(zx) = \frac{1}{1 - zx} = 1 + zx + z^2x^2 + \cdots,$$

* Cf. Le Roy, *loc. cit.*, pp. 414–415.

† *Loc. cit.*

so that

(14) $$a_n = \int_0^\infty f(z)\cdot z^n dz.$$

At first sight this choice of functions would seem to be a very desirable one, for the function defined by the divergent series is obtained in the familiar form

(15) $$\phi(z) = \int_0^\infty \frac{f(z)dz}{1 - zx}.$$

Upon examination, however, it turns out to be otherwise. For suppose the divergent series to be given and $f(z)$ is to be found. The problem is then a very difficult one, that of the inversion of the integral (14) when a_n is given for all values of n. This is what *Stieltjes* terms "the problem of the moments." It does not admit of a unique solution, for *Stieltjes* himself* gives a function,

$$f(z) = e^{-\sqrt[4]{z}} \sin \sqrt[4]{z},$$

which will make $a_n = 0$ for all values of n. If the supplementary condition is imposed that $f(z)$ shall not be negative between the limits of integration, only a single solution $f(z)$ is possible, but the divergent series is thereby restricted to belong to that class which *Stieltjes* derives naturally and elegantly by the consideration of his continued fraction.

Thus far our attention has been confined exclusively to integrals in which one of the limits of integration is infinite. There are, however, advantages in using appropriate integrals having both limits finite, at least if the given series is convergent and the integral is used for the purpose of analytic continuation. In particular, the integral

(16) $$f(x) = \int_0^1 V(z)F(zx)dz$$

should be noted, to which *Hadamard* has drawn attention in his thesis.† This falls under *Vallée-Poussin's* theorem when $V(z)$ is

* *Loc. cit.*, § 55.

† *Journn. de Math.*, ser. 4, vol. 8 (1892), pp. 158–160.

continuous along the path of integration and when also $F(u)$ is analytic in $u = zx$ for all values of z upon the path of integration and for values of x in some specified region of the x-plane. If, as we suppose, the path is rectilinear, the values of x to be excluded are evidently those which lie on the prolongations of the vectors from the origin to the singular points of $F(x)$. The region of convergence of (16) is consequently a star, whose boundary consists of prolongation of these vectors.* Thus *Hadamard's* integral, when applied to the analytic continuation of a function, is superior to *Borel's* in the extent of its "region of summability." This is illustrated in *Le Roy's* thesis † with the very familiar series :

$$1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot \cdots \cdot 2n} x^n.$$

Here the coefficient of x^n is

$$\frac{1}{\pi} \int_0^1 \frac{z^n dz}{\sqrt{z(1-z)}},$$

so that

$$f(x) = \frac{1}{\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}(1-zx)}.$$

Since $F(zx) \equiv 1/(1-zx)$, the region of summability is the entire plane of x with the exception of the part of the real axis between $x = 1$ and $x = \infty$. *Borel's* polygon of summability for the series, on the other hand, is only the half plane lying to the left of a perpendicular to the real axis through the point $x = 1$.

Much, it seems to me, can yet be done in following up the use of *Hadamard's* integral. One special case has been studied already by *Le Roy*, in which the $(n+1)$th coefficient of (I) has the form

$$a_n = \int_0^1 z^n \phi(z) dz.$$

---

* This conclusion also holds if only $\int_0^1 V(z)dz$ is an absolutely convergent integral, as is shown by Hadamard.

† p. 411.

The series therefore defines a function

$$\int_0^1 \frac{\phi(z)dz}{1 - zx}$$

which is analytic over the entire plane except along the real axis between $x = 1$ and $x = \infty$. The path of integration may also permit of deformation so as to show that the cut between the points is not an essential cut. It is interesting to note that if $\phi(z)$ is positive between 0 and 1, the primary branch of the function has only real roots which are, moreover, greater than 1.*

LECTURE 3. *On the Determination of the Singularities of Functions Defined by Power Series.*

Up to the present time comparatively little successful work has been done in determining the singularities of functions defined by power series, and the little which has been done relates mostly to singularities upon the circle of convergence. Work of this special nature I shall omit from consideration here, thus passing over the memoirs of *Fabry*, and I shall call your attention to the literature which treats of the singularities in a wider domain.

The most fundamental and practical result yet obtained is undoubtedly a brilliant theorem of *Hadamard*,† in the wake of which a number of other interesting memoirs have followed. This theorem is as follows:

If two analytic functions are defined by the convergent power series

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \tag{1}$$

$$\psi(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \tag{2}$$

the only singularities of the function

$$f(x) = a_0 b_0 + a_1 b_1 x + a_2 b_2 x^2 + \cdots \tag{3}$$

will be points whose affixes γ_{ij} are the product of affixes of the singular points α_i and β_j of the first two functions.

---

* Le Roy, *loc. cit.*, pp. 330–331.

† *Acta Math.*, vol. 22 (1898), p. 55.

The possibility that $x = 0$ should, in addition, be a singular point has been pointed out since by *Lindelöf.*

Although *Hadamard's* proof of the theorem is not a complicated one, I shall present here a still simpler proof given by *Borel.** Let R and R' be the radii of convergence of (1) and (2) respectively, and take a number ρ such that $R/\rho > 1/R'$. If then $|zx| \leqq |\rho x| < R$ and $|x| > 1/R'$, the product of $\phi(zx)$ and $\psi(1/x)$ can be developed into a *Laurent's* power series which is valid in a circular ring in the x-plane, having its center at the origin and the outer and inner radii R/ρ and $1/R'$ respectively. In this product the absolute term is obviously

$$f(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots. \tag{4}$$

Consider now the integral

$$\frac{1}{2i\pi} \int_c \frac{\phi(zx)\psi\left(\frac{1}{x}\right) dx}{x}, \tag{5}$$

in which c is a closed path surrounding the origin and contained within the circular ring. As long as z in its plane lies within a circle of radius $\rho < RR'$, having its center in the origin, the integral will surely define a function of z, and this function is evidently equal to the residue of the integrand for $x = 0$, which is $f(z)$.

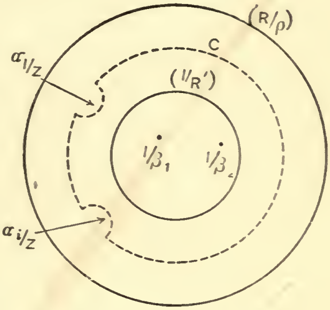
We shall now seek to extend this function by varying z and at the same time deforming appropriately the path of integration. By the theorem of *Vallée Poussin* quoted in Lecture 2, the integral will continue to represent an analytic function of z, provided at every stage the integrand remains analytic in x and z; x being any point upon the path of integration. Now the values to be avoided are clearly the singular points of the functions $\phi(zx)$ and $\psi(1/x)$; namely the points:

* *Bull. de la Soc. Math. de France,* vol. 26 (1898), pp. 238–248.

An interesting proof "*in multi case*" is given without the use of integrals by Pincherle in the *Rendiconto della R. Accad. delle Scienze di Bologna,* new ser., vol. 3 (1898–9), pp. 67–74.

$$zx = \alpha_i, \quad x = \frac{1}{\beta_j}.$$

The points $x = 1/\beta_j$ lie within the circle $(1/R')$ which is the inner circumference of the ring, while the points $x = \alpha_i/z$ before the variation of z lie without the outer circumference (R/ρ). For simplicity of presentation it may be convenient to assume at first that these points form an aggregate of isolated points. Suppose then that z follows any path in its plane emerging from the circle (ρ). Then the points α_i/z describe certain corresponding paths which we will mark in the x-plane. At the same time the contour c may be deformed continuously so as to recede before the points α_i/z without sweeping over any point $1/\beta_j$, provided merely that α_i/z never collides with a point $1/\beta_j$; that is, z must never pass through a point $\alpha_i\beta_j$. Now when z is held fixed, a deformation in the contour c, subject of course to the condition indicated, produces no change in the value of the integral $f(z)$, since the integrand is holomorphic between the initial and deformed paths. On the other hand, when the path is kept fixed and z is varied, we have the analytic continuation of $f(z)$ in accordance with the theorem of *Vallée Poussin*. By the two changes together $f(z)$ may be continued over the entire plane of z with the exception of the points $\alpha_i\beta_j = \gamma_{ij}$. To these should, of course, be added $z = \infty$, also $z = 0$ as a possible singular point for any branch of $f(z)$ except the initial branch.



It should be observed that γ_{ij} is shown to be a potential rather than an actual singular point. When, however, it is such a point, the character of the point depends in general solely upon the nature of the singularities α_i and β_j for (1) and (2) respectively. This fact was noticed by *Borel* and demonstrated in the following manner. Let

$$c_0 + c_1x + c_2x^2 + \cdots$$

be any convergent series defining a function $\phi_1(x)$ which is regular at α_i. Then $\phi_2(x) = \phi_1(x) + \phi(x)$ is a function which has at α_i the same singularity as $\phi(x)$. The combination of the series for $\phi_2(x)$ and for $\psi(x)$ by *Hadamard's* process gives the function

$$f_2(x) = (a_0 + c_0)b_0 + (a_1+c_1)b_1x + (a_2+c_2)b_2x^2 + \cdots = f(x) + f_1(x),$$

in which

$$f_1(x) = c_0b_0 + c_1b_1x + c_2b_2x^2 + \cdots.$$

Now since $\phi_1(x)$ is regular at α_i, when compounded with $\psi(x)$ it must give a function $f_1(x)$ which is regular at γ_{ij}. It follows that $f_2(x)$ and $f(x)$ have the same singularity at γ_{ij}. Thus the nature of this singular point is not altered by any change in $\phi(x)$ or $\psi(x)$ which does not affect the character of the points α_i and β_j. It depends therefore solely upon the character of the singularities compounded.

Complications arise only when there is a second pair of singularities α_k, β_l such that

$$\gamma_{ij} = \alpha_i\beta_j = \alpha_k\beta_l.$$

Clearly the resultant singularity is then dependent upon both pairs. Their effects may be so superimposed as to create an ugly singularity, or they may, on the other hand, so neutralize each other that γ_{ij} is a regular point. Very simple examples of the latter occurrence can be easily given. It seems probable that when γ_{ij} is but once a product of an α by a β, it must always be a singular point, but this has not yet been proved. Its demonstration will greatly enhance the value and applicability of *Hadamard's* theorem, for then it can be stated in numerous cases, not what the singular points of $f(x)$ may be, but what they actually are.

A detailed study of the nature of the dependence of the singularity γ_{ij} upon α_i and β_j would probably be both interesting and profitable. *Borel* examines the case in which α_i and β_j are poles of any orders, p and q, and shows that γ_{ij} is then a pole of order $p + q - 1$. It can, furthermore, be easily shown that whenever α_i is a pole of the first order, γ_{ij} is the same kind of singular point as β_j. For suppose that we put $\alpha_i = 1$, which may be done without loss of generality. The principal part of $\phi(x)$ at the pole α_i is then

$$\frac{A_i}{x-1} = -A_i(1 + x + x^2 + \cdots),$$

and the composition of this with $\psi(x)$ gives for the corresponding component of $f(x)$

$$-A_i\,(b_0 + b_1x + b_2x^2 + \cdots).$$

Hence the singularities γ_{ij} and β_j differ by a multiplicative constant.

Only one other general fact concerning the composition of singularities seems to be known. *Borel* proves that if the functions $\phi(x)$ and $\psi(x)$ are one-valued at α_i and β_j respectively, $f(x)$ is also one-valued at γ_{ij}. Thus when two one-valued functions are compounded, the resultant function is also one-valued. But this statement, as he himself points out, must be correctly construed and will not necessarily hold true when the singular points of the two given functions are not sets of isolated points but condense in infinite number along curves. To construct an example in which $f(x)$ in not one-valued, *Borel* makes use of the fact, now so well known, that the decision whether the circle of convergence is or is not a natural boundary of a given series depends upon the arguments of its coefficients. If, for instance, we take the series

$$1 + e^{i\theta_1}x + e^{i\theta_2}x^2 + \cdots,$$

which has a radius of convergence equal to 1, by a proper choice of the arguments θ_n the circle of convergence can be made a natural boundary. Put now

$$\sqrt{1-x} = c_0 + c_1x + c_2x^2 + \cdots, \tag{6}$$

in which the coefficients are necessarily real. Clearly the unit circle will be a natural boundary for

$$\phi(x) = c_0 + c_1e^{i\theta_1}x + c_2e^{i\theta_2}x^2 + \cdots.$$

and for

$$\psi(x) = 1 + e^{-i\theta_1}x + e^{-i\theta_2}x^2 + \cdots.$$

Yet the function $f(x)$ which is derived from these two one-valued functions by *Hadamard's* process is the two-valued function (6) which exists over the entire plane of x.

I have dwelt at some length upon *Hadamard's* theorem and its consequences because of their evident interest and importance. It is worthy of note that for analytic functions defined by power series the first great advance in the determination of the singularities over their entire domain has been made by methods that are roughly parallel to those currently employed in the consideration of their convergence. The convergence of series is indeed too difficult a question to be settled by any one rule or by any finite set of rules, but the methods of comparison with series known to be convergent have been found to be not only most efficient but also adequate for most practical purposes. In somewhat similar fashion *Hadamard's* theorem will determine the singular points of numerous functions by linking them with other series, of which the singularities are known.

One of the simplest applications of this theorem is obtained by compounding a given series

$$(7) \qquad a_0 + a_1x + a_2x^2 + \cdots$$

with itself once, twice, $\cdots$, to m times. All the singularities of the resulting series

$$(8) \qquad a_0^i + a_1^ix + a_2^ix^2 + \cdots \qquad (i = 1, 2, \cdots, m),$$

except possibly $x = 0$ and $x = \infty$, are included among the points obtained by multiplying i affixes of the singular points of (7) among themselves in all possible ways. If the m series (8) are multiplied each by a constant k_i and are then added, a new series

$$(9) \qquad G(a_0) + G(a_1)x + G(a_2)x^2 + \cdots$$

is obtained, in which $G(u)$ denotes the polynomial $k_1u + \cdots + k_mu^m$.

This function has no singular points other than those which are possible for the m series from which it was derived. When r different series

$$a_0 + a_1x + a_2x^2 + \cdots,$$

$$b_0 + b_1x + b_2x^2 + \cdots,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$r_0 + r_1x + r_2x^2 + \cdots,$$

are used, a similar conclusion is reached for the series

$$G(a_0, b_0, \cdots, r_0) + G(a_1, b_1 \cdots, r_1)x + G(a_2, b_2, \cdots, r_2)x^2 + \cdots,$$

where G denotes a polynomial in which the constant term is lacking.

These results are of particular interest when applied to the series

$$(10) \qquad 1 + x + 2x^2 + \cdots + nx^n + \cdots$$

and

$$(11) \qquad 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots,$$

which are the expansions of $1 + x/(1 - x)^2$ and $\log(1 + x)$. Since these functions have only one singular point, $x = 1$, in the finite plane, the only possible singularities of

$$\Sigma\, G\left(n, \frac{1}{n}\right)x^n$$

are $x = 0, 1, \infty$.*

The continued repetition of the above process for combining series leads naturally to a consideration of series of the form

$$(12) \qquad \Sigma P(a_n)x^n$$

in which a convergent power series $P(u)$ appears in place of the polynomial $G(u)$. Various theorems concerning cases of this

* Obviously a constant term can be included now in the polynomial $G(n, 1/n)$.

series have been given recently by *Leau*,* *Le Roy*,† *Desaint*,‡ *Lindelöf*,§ *Ford* || and *Faber*,¶ though the proof of some of these theorems has no direct relation to *Hadamard's* theorem. The importance of such work is, however, apparent, inasmuch as numerous series which occur in analysis can be put into the form under consideration, as for example $\Sigma(\sin \pi/n)\, x^n$.

Three cases must be distinguished according as the radius of convergence of the initial series (7) is less than, equal to, or greater than 1. If the radius is less than 1, the singular point nearest to the origin has a modulus less than 1, and the continued multiplication of the affix of the point by itself gives a series of points which approach indefinitely close to the origin. The presumption, therefore, would naturally be that the series (12) is then divergent, but this is very far from being always true, as will be seen at once by referring to the series $\Sigma(x^n \sin a_n)$ and $\Sigma(x^n \cos a_n)$ in which a_n is real. The applicability of *Hadamard's* theorem consequently ceases.

The case in which the radius of convergence of (7) is greater than 1 has been investigated very recently by *Desaint*. In this case the expected theorem is obtained. If, namely, $P(u)$ is a convergent series without a constant term, $\Sigma P(a_n)x^n$ defines a function which can have no singular points, besides $x = 0$ and $x = \infty$, than those which result from the multiplication of the affixes of the singular points among themselves in all possible ways and to any number of times.** *Desaint's* proof is based upon the fact that $\Sigma P(a_n)z^n$, after the omission of a suitable number of terms, can be expressed in the form

---

* *Journ. de Math.*, ser. 5, vol. 5 (1899), p. 365.

† *Loc. cit.*

‡ *Journ. de Math.*, ser. 5, vol. 8 (1902), p. 433.

§ *Acta Societatis Scientiarum Fennicæ*, vol. 31 (1902).

|| *Journ. de Math.*, ser. 5, vol. 9 (1903), p. 223.

¶ *Math. Ann.*, vol. 57 (1903), p. 369.

** This is a somewhat sharper statement of the result than that given by Desaint. In his theorem $x = 1$ is given as a possible singular point, but this, as appears from the proof to be given here, is due solely to the admission of a constant term into $P(u)$. He also fails to note that $x = 0$ may be a singular point.

$$z^N \sum_{k=N}^{\infty} c_n \int_{c'} \cdots \int_{c'} \frac{\dfrac{f(t_1) f(t_2) \cdots f(t_k)}{(2i\pi)^k (t_1 t_2 \cdots t_k)^{N+1}}}{1 - \dfrac{z}{t_1 t_2 \cdots t_k}} dt_1 dt_2 \cdots dt_k,$$

in which $f(t)$ is the function defined by (7) for $x = t$, c' is an appropriately chosen contour, and c_n denotes the nth coefficient of

$$P(u) = c_1 u + c_2 u^2 + \cdots.$$

Although his proof is essentially simple in character, I shall give here a new and simpler proof, based directly upon *Hadamard's* theorem.

Place first

$$f_i(x) = a_0^i + a_1^i x + a_2^i x^2 + \cdots \qquad (i = 2, 3, \cdots),$$

and consider the expression

$$c_n . f_n(x) + c_{n+1} . f_{n+1}(x) + \cdots$$

in which n denotes some fixed integer. If $r > 1$ denotes the radius of convergence of the fundamental series (7), the radius of $f_i(x)$ will be r^i. Describe about the origin a circle (r') having a radius $r' < r^n$. If a sufficient number of initial terms be cut off in each of the series,

$$f_n(x), \quad f_{n+1}(x), \quad \cdots, \quad f_{2n}(x),$$

the maximum absolute values of the remainders within or upon the circle (r') can be made as small as is desired. Suppose then that after m terms of each have been removed, the remainders

$$(13) \qquad r_n(x), \quad r_{n+1}(x), \quad \cdots, \quad r_{2n}(x)$$

do not exceed

$$\epsilon^n, \quad \epsilon^{n+1}, \quad \cdots, \quad \epsilon^{2n}$$

respectively, in which ϵ is some small positive number. Let us now substitute in *Hadamard's* integral

$$F(z) = \frac{1}{2i\pi}\int_c \frac{\phi(zx)\psi\left(\frac{1}{x}\right)dx}{x},$$

any two of the functions (13) for ϕ and ψ.

Put for example

$$\phi(zx) = r_{n+i}(zx), \qquad \psi\left(\frac{1}{x}\right) = r_{n+j}\left(\frac{1}{x}\right),$$

and choose the unit circle as the path of integration. Then if $|z| \leqq r'$, the absolute values of the arguments of the series $\phi(zx)$ and $\psi(1/x)$ will be less than their radii of convergence since $|x| = 1$ and $r > 1$. The conditions for the existence of *Hadamard's* integral are therefore fulfilled. Since also

$$|r_{n+i}(zx)| < \epsilon^{n+i}, \qquad \left|r_{n+j}\left(\frac{1}{x}\right)\right| < \epsilon^{n+j},$$

we have

$$|F(z)| \leqq \frac{\epsilon^{2n+i+j}}{2\pi}\int\frac{|dx|}{|x|} = \epsilon^{2n+i+j}.$$

But by *Hadamard's* theorem $F(z) = r_{2n+i+j}(z)$, and hence

$$(14) \qquad |r_i(z)| < \epsilon^i \qquad (|z| \leqq r'),$$

for all values of i from $2n$ to $4n$ inclusive. The reasoning can now be repeated with $2n$ in place of n, and so on; therefore (14) is true for all values of $i \geqq n$.

Thus far the value of ϵ has remained arbitrary. Let its value now be taken less than the radius of convergence of $P(u)$. Then by (14) the series

$$(15) \qquad c_n r_n(x) + c_{n+1}r_{n+1}(x) + \cdots$$

will be uniformly convergent in (r'). Since, furthermore, all the component series $r_{n+i}(i = 0, 1, 2, \cdots)$ are likewise so convergent, by a fundamental and familiar theorem of *Weierstrass** the terms of the collective series (15) may be rearranged into an ordinary series in ascending powers of x. But this rearrangement gives

* Harkness and Morley's *Introduction to the Theory of Analytic Functions*, p. 134.

$$\sum_{j=m}^{\infty}\left(\sum_{i=n}^{\infty} c_i a_j^i\right)x^j,$$

or the remainder after the $(m-1)$th power of x in

$$(16) \qquad \sum_{j=0}^{\infty} P(a_j)x^j - c_1 f(x) - c_2 f_2(x) - \cdots - c_{n-1} f_{n-1}(x).$$

Now the series (15) before its rearrangement was a uniformly convergent series of analytic functions and defined a function which was analytic within (r'). It follows that (16) is also analytic within this circle, and hence

$$\sum_{n=1}^{\infty} P(a_n)x^n$$

has no singularities within this circle except those of

$$f_1(x),\ f_2(x),\ \cdots,\ f_{n-1}(x).$$

But the radius of (r') was any quantity short of r^n, and this conclusion therefore holds within a circle having its center in the origin and a radius equal to r^n. By increasing n indefinitely, the theorem of *Desaint* results. It is evident also that if $f_1(x)$, and therefore $f_i(x)$, represents a one-valued function, $\Sigma P(a_n)x^n$ must also be such a function.

There remains yet for consideration the third class of cases in which the radius of convergence of the fundamental series is 1. If upon the circle of convergence there is any singular point with an incommensurable argument, the continued multiplication of its affix by itself gives a set of points everywhere dense upon the circle of convergence. It is therefore to be expected that this circle will be, in general, a natural boundary for $\Sigma P(a_n)x^n$, and accordingly the cases which will be of chief interest will be those in which all the singular points upon the circle have commensurable arguments. A simple case of this character is obtained when either (10) or (11) is chosen as the generating series. If the former be selected, the resulting series has the form $\Sigma P(n)x^n$. This has a special interest inasmuch as its study has proved to be of profit both for the theory of analytic continuation and of divergent series. The reason becomes apparent when the statement is made that it is possible to throw any Taylor's series, $\Sigma a_n x^n$, whether convergent or divergent, into the particular form $\Sigma P(n)x^n$, and in an infinite number of ways. This fact follows as a corollary of a very general theorem of *Mittag-Leffler*,* which, when restricted to the special case before us, establishes the existence of a function $P(x)$, which is holomorphic over the entire finite plane and assumes the pre-assigned values $a_0, a_1, a_2, \cdots$ in the points $x = 0, 1, 2, \cdots$. Consequently the character of the function defined by $\Sigma P(n)x^n$ is made to depend upon the behavior of $P(x)$ as x approaches ∞.

Inasmuch as $\Sigma P(n)x^n$ is perfectly general, limitations must be imposed upon $P(u)$ in any attempt to extend *Hadamard's* theorem to this series. But whenever the theorem is applicable, the only possible singularities of $\Sigma P(n)x^n$ are $x = 0, 1, \infty$. *Leau* † establishes the correctness of this result when $P(u)$ is an entire function of order less than 1,‡ giving also a more general theorem § concerning $\Sigma P(a_n)x^n$ of which this is a special case. The like conclusion holds concerning the singular points of $\Sigma P(1/n)x^n$, provided only that $P(x)$ is holomorphic at the origin.‖

Very recently these results of *Leau* have been proved more simply by *Faber*, but in a more restricted form, an artificial cut being drawn from $x = 1$ to $x = \infty$ to obtain a one valued function. In addition, *Faber* shows that if for any prescribed ϵ and for a sufficiently large r the inequality

$$(17) \qquad |P(re^{i\phi})| < e^{\epsilon r}$$

---

* *Acta Math.*, vol. 4 (1884), p. 53, theorem D. For a reference to this theorem I am indebted to Professor Osgood. Theorem 2 of Desaint's memoir (p. 438) is in contradiction with this, but his proof is here inadequate since r_k (p. 440) has not necessarily a lower limit.

† *Loc. cit.*, p. 418.

‡ He also shows that $\Sigma P(n)x^n$ is then a one-valued function.

§ *Loc. cit.*, p. 417. See also *Bull. de la Soc. Math. de France*, vol. 26 (1898), p. 267.

‖ *Loc. cit.*, p. 418; see also p. 407.

is fulfilled, the point $x = 1$ must be an essential singularity, and the function represented by $\Sigma P(n)x^n$ is consequently one-valued.* Conversely, if $f(x)$ is a one-valued function which has only one singular point, and if that point is an essential singularity, $f(x)$ can be expressed in the form $\Sigma P(n)x^n$, in which $P(u)$ is an entire function satisfying (17). More generally, if there are l essential singularities $x_1, \cdots, x_l$ and no other singular points in the finite plane, the coefficient of x^n must be

$$a_n = \frac{1}{x_1^n} P_1(n) + \cdots + \frac{1}{x_l^n} P_l(n) + a'_n,$$

in which $P_1(n), \cdots, P_l(n)$ are entire functions of the nature above specified and in which $\lim \sqrt[n]{a'_n} = 0$. This converse has an especial interest because as yet few theorems have been discovered giving the necessary form of the coefficients of a power series for an analytic function with prescribed functional properties.

Other theorems concerning $\Sigma P(n)x^n$ have recently been derived without requiring that $P(n)$ shall be holomorphic over the entire plane.

As a sample of these I shall cite in conclusion the following theorem of *Lindelöf*: †

If $P(n)$ represents a function fulfilling the following conditions:

1. $P(z)$ is analytic for every point of the complex plane $z = \tau + it$ for which $\tau \geqq 0$ (except possibly at the origin, for which $P(z)$ has a determinate value).

2. A number ϵ being arbitrarily given, it is possible to find another number R such that by putting $z = re^{i\phi}$ we will have for $r > R$

$$|P(z)| < e^{\epsilon r} \qquad \left(-\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2}\right):$$

---

* Le Roy three years earlier had noted this conclusion when $P(x)$ is an entire function whose "apparent order" is less than 1; *loc. cit.*, p. 348, footnote. Faber does not seem to be aware of Le Roy's statement. The difference between the two statements is slight but becomes important in formulating the new and interesting converse which Faber adds.

† *Loc. cit.*, § 13.

then the principal branch of the function $\Sigma P(n)z^n$ will be holomorphic throughout the complex plane excepting possibly on the segment $(1, +\infty)$ of the real axis. Furthermore, the function approaches 0 as a limit when x tends toward the point at infinity along any ray having an argument between 0 and 2π.

LECTURE IV. *On Series of Polynomials and of Rational Fractions.*

In the last two lectures I have spoken of the use of integrals for the study of analytic extension and of divergent series. The topic of to-day's lecture is the representation of functions by means of series of polynomials and of rational fractions. This subject forms a very natural transition to the succeeding lecture upon continued fractions, since an algebraic continued fraction is in reality nothing but a series of rational fractions advantageously chosen for the study of a corresponding function which, when known, is commonly given in the form of a power series.

The literature relating directly or indirectly to series of polynomials and of rational fractions is a vast one, with many ramifications. Thus in one direction there are various researches of importance upon the non-uniform convergence of series of continuous functions, and in this connection I may refer particularly to the recent work of *Osgood* and *Baire*, an excellent report of which is contained in *Schönflies' Bericht über die Mengenlehre.** Another part of the field comprises numerous memoirs devoted to special series of polynomials and rational fractions. Quite recently a more systematic and general study has been begun by *Borel*, *Mittag-Leffler*, and others, and it is to this that I am to call your especial attention.

Two very familiar facts, both discovered by *Weierstrass*, may be said to be the origin of this study. I refer, of course, to the theorem that any function which is continuous in a given finite interval of the real axis can be expressed in that interval as an

* *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 8, pp. 224–241.

absolutely and uniformly convergent series of polynomials,* and, secondly, to the possibility that a single series of rational fractions may represent two or more distinct analytic functions in different portions of its domain of convergence. A notable advance upon the theorem first mentioned was made by *Runge*† in 1884, who proved that any one-valued analytic function throughout the domain of its existence can be represented by a series of rational functions; furthermore, this domain may be of any shape whatsoever, provided only it forms a two-dimensional continuum. *Runge's* proof of these important results is not only worthy of careful study, but contains also certain conclusions which were announced again by *Painlevé*‡ in 1898, though without proof. The conclusions reached were as follows :

Let D be a domain consisting of any number of separate pieces of the complex plane, in each of which we will suppose an analytic function to be defined. The functions thus defined can be, at pleasure, either distinct functions or parts of one or more functions. In any case a series of rational functions can be formed which will converge absolutely and uniformly in any region lying in the interior of D and represent in each separate piece the prescribed function. Furthermore, this representation can be made in an infinite number of ways. Let the ensemble of the points excluded from D be represented by E. When E consists of a single connected continuum of any sort, whether linear or areal, any point a of E can be arbitrarily selected, and the function can be expanded into the series

$$\sum_0^\infty G_n\left(\frac{1}{x-a}\right)$$

* "Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen"; *Berliner Sitzungsberichte*, 1885, p. 633 or *Werke*, vol. 3, p. 1. Simple proofs of the theorem have been given by Lebesque, *Bull. des Sciences Math.*, ser. 2, vol. 22 (1898), p. 278, and by Mittag-Leffler, *Rendiconti di Palermo*, vol. 14 (1900), p. 217, with an extension to functions of two variables. In this connection see Painlevé's note in the *Compt. Rend.*, vol. 126 (1898), p. 459.

† *Acta Math.*, vol. 6, p. 229.

‡ *Compt. Rend.*, vol. 126, pp. 201 and 318.

in which $G_n[1/(x-a)]$ denotes a polynomial in $1/(x-a)$. If, in particular, the continuum E contains the point $x=\infty$, an ordinary series of polynomials, $\Sigma G_n(x)$, can be employed. When E consists of a finite number of separate pieces (or isolated points), the expansion can be put under the form

$$\sum_{n=1}^{\infty} G_n^{(1)}\left(\frac{1}{x-a_1}\right)+\sum_{n=1}^{\infty} G_n^{(2)}\left(\frac{1}{x-a_2}\right)+\cdots+\sum_{n=1}^{\infty} G_n^{(q)}\left(\frac{1}{x-a_q}\right),$$

in which $a_1, \cdots, a_q$ are points arbitrarily chosen in the separate pieces.

In the familiar case in which only a single analytic function

$$(1) \qquad a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

is given, it is natural to seek a series of polynomials having the greatest possible domain of convergence. Unless the function is one-valued, the most convenient domain is in general the star of *Mittag-Leffler*. This is constructed for the series (1) by first marking on each ray which terminates in a the nearest singular point and then obliterating the portion of the ray beyond this point. The region which remains when this has been done is a star having a for its center. *Mittag-Leffler* * shows that within the star the given analytic function can be represented by a series of polynomials in which the coefficients of the polynomials depend only upon the value of the function and its derivatives at a, † or, in other words, upon the coefficients of (1). If, in short, we put

$$g_n(x)=\sum_{\lambda_1=0}^{n^2}\sum_{\lambda_2=0}^{n^4}\cdots\sum_{\lambda_n=0}^{n^{2n}}\frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)!\,a_n}{\lambda_1!\,\lambda_2!\cdots\lambda_n!}\left(\frac{x-a}{n}\right)^{\lambda_1+\lambda_2+\cdots+\lambda_n}$$

and

$$G_n(x)=g_n(x)-g_{n-1}(x),$$

* *Acta Math.*, vol. 23 (1899), p. 43; vol. 24, pp. 183, 205; vol. 26, p. 353. A good summary is found in the *Proc. of the London Math. Soc.*, vol. 32 (1900), pp. 72–78.

† In this respect his work is superior to that of Runge and others. Runge, for example, presupposes a knowledge of the function at an infinite number of points.

then $\sum_{n=0}^{\infty} G_n(x)$ is a series which converges uniformly in any region lying, with its boundary, entirely in the interior of the star. The series may also converge outside the star. *Borel* * has shown, furthermore, that the series of *Mittag-Leffler* is not the only possible one, but there is an infinity of polynomial series sharing the same property within the star.

It will be noticed that the construction of the series of *Mittag-Leffler* is in no wise dependent upon the convergence of the initial power series. In certain cases, at least, the polynomial series converges when the given series (1) is itself divergent. It is natural therefore to look for a theory of divergent series based upon convergent series of polynomials. As yet, however, no such theory has been invented. One of the chief difficulties in the way is that the polynomial series do not afford a unique mode of representing an analytic function. Now the difference between any two series of polynomials for the same function in an assigned area is a third series which vanishes at every point of the area, though the separate terms do not. This is a decidedly awkward point, and occasions difficulty in proving or disproving the identity of two functions expressed by polynomial series. It is true, indeed, that this difficulty will scarcely present itself when we start with a convergent power series which is to be continued analytically, the polynomial series then giving continuations of a common function. But when the series (1) is divergent and there is no known function which it represents, it is an open question whether the different series of polynomials which are obtained from (1) by application of diverse laws will furnish the same or different functions. If different functions, is there any ground for preferring one series of polynomials to another?

Up to the present time two essentially different principles seem to have been followed in the formation of series of polynomials. In the work of *Runge, Borel, Painlevé* and *Mittag-Leffler* the coefficients in the polynomials vary with the character of the ana-

* *Ann. de l'Ec. Nor.*, ser. 2, vol. 16 (1899), p. 132, or *Les Séries divergentes*, p. 171.

lytic function to be represented; for example, in the polynomials of *Mittag-Leffler* they are functions of the coefficients of the given element, $\sum a_n x^n$. By appropriately choosing the coefficients of the polynomials these writers obtain a very large region of convergence and at the same time are able to greatly vary its shape. On the other hand, the series which are met in the practical branches of mathematics—for instance, in the theory of zonal harmonics—have the form

$$(2) \qquad c_0 G_0(x) + c_1 G_1(x) + c_2 G_2(x) + \cdots$$

in which the polynomials $G_n(x)$ are entirely independent of the function represented, while the c_i vary. The polynomials themselves are selected according to the shape of the region of convergence. Thus if the region is a circle, we may put

$$G_n(x) = (x - a)^n,$$

and we have then the ordinary Taylor's series. Or if it be an ellipse having the foci ± 1, we may take for our polynomials either the successive zonal harmonics or a second succession of polynomials (also called *Legendre's* polynomials) which are connected by the recurrent relation:

$$(3) \qquad G_{n+2}(x) - 2x(2n+3)G_{n+1}(x) + 4(n+1)^2 G_n(x) = 0.$$

In a recent number of the *Mathematische Annalen* (July, 1903) *Faber* has considered this second class of polynomials from a somewhat general point of view and has demonstrated that any function which is holomorphic within a closed branch of a single analytic curve, as for example an ellipse or a lemniscate of one oval, can be expressed by a series of the form (2). The properties of his series are similar to those of Taylor's series. In the case of the latter, to ascertain whether $\Sigma a_n x^n$ converges in the interior of a circle having its center in the origin and a radius R, we have only to determine the maximum modulus of a point of condensation of the set of points $\sqrt[n]{a_n}$ $(n = 1, 2, 3, \cdots)$. If it is exactly equal to $1/R$, the circle (R) is the circle of convergence, and there is at least one singularity upon its circumference. If, on the other hand, it is greater or less than $1/R$, the series will have a smaller or a larger circle of convergence. So also to the given branch of the analytic curve there corresponds a certain critical value. When this is exactly equal to the upper limit of $|\sqrt[n]{c_n}|$ in *Faber's* series, the given analytic branch is the curve of convergence. At every point within, the series converges, while it diverges at every exterior point, and upon the curve there must lie at least one singular point of the function defined by (2). If, however, the upper limit is greater or less than the critical value, we consider a certain series of simple, closed analytic curves, (as for example a series of confocal ellipses), among which the given analytic branch must, of course, be included. The curve of convergence is then fixed by the reciprocal of the upper limit of $|\sqrt[n]{c_n}|$ provided this limit is not too large. Moreover, as in the case of Taylor's series, the function cannot vanish identically unless every $c_n = 0$, and in consequence the series vanishes identically. It is therefore impossible that the same function shall be represented by two different series of the given form.

In view of the last mentioned fact it might be of especial interest to apply this class of polynomial series to the study of divergent series.

In the most familiar and useful polynomial series the successive polynomials are connected by a linear law of recurrence,

$$(4) \qquad k_0 G_{n+m}(x) + k_1 G_{n+m-1}(x) + \cdots + k_m G_n(x) = 0,$$

in which the coefficients k_i are polynomials in x and n. Thus the zonal harmonics have as their law of recurrence

$$(n+1)G_{n+1}(x) - (2n+1)xG_n(x) + nG_{n-1}(x) = 0.$$

Many series of this nature are also included in the class considered by *Faber*. The form of the region of convergence has been determined by *Poincaré* * upon the hypothesis that equation

* *Amer. Journ. of Math.*, vol. 7 (1885), p. 243.

(4) has a limiting form for $n = \infty$. Let the equation be first divided through by k_0, and then denote the limits of the successive coefficients for $n = \infty$ by $k_1(x), k_2(x), \cdots k_m(x)$. Construct next the auxiliary equation

$$(5) \qquad z^m + k_1(x)z^{m-1} + k_2(x)z^{m-2} + \cdots + k_m(x) = 0.$$

Except for particular values of x there will be one root of this equation which has a larger modulus than any other. Let $r(x)$ be that root. *Poincaré** shows that with increasing n the ratio $G_n(x)/G_{n-1}(x)$ will approach, in general, this root as its limit. The region of convergence is therefore confined by a curve of the form $C = |r(x)|$, and the value of C for the series (2) is to be taken equal to the radius of convergence of $\Sigma c_n y^n$.†

By way of illustration let us take the series $\Sigma c_n G_n(x)$ in which the polynomial obeys the law

---

* More specifically, Poincaré proves that if no two roots of (5) are of equal modulus, $G_n(x)/G_{n-1}(x)$ has always a limit, and this limit is equal to some root of (5), usually the one of greatest modulus.

† Poincaré has given no proof that the series (2) will converge at those points within the curve $|r(x)| = C$, for which there are two or more distinct roots of (5) having a common modulus greater than the moduli of the remaining roots. Thus in the example which is quoted below (p. 127), these are the points of the real axis which are included between $+1$ and -1. This gap in Poincaré's theory can be filled in by the following theorem which I have given in the *Transactions of the Amer. Math. Soc.*, vol. 1 (1900), p. 298: If the coefficients in the series $\Sigma A_n y^n$ are connected by a recurrent relation having the limiting form

$$A_n + k_1 A_{n-1} + \cdots + k_m A_{n-m} = 0,$$

the series will converge at the worst within a circle whose radius is the reciprocal of the greatest modulus of any root of the auxiliary equation

$$z^n + k_1 z^{n-1} + \cdots + k_m = 0.$$

Denote this maximum by r, *irrespective of the number of roots having this maximum modulus.* Then

$$|A_n| < M(r+\varepsilon)^n \qquad (n = 1, 2, \ldots).$$

Hence if C is the radius of convergence of $\Sigma c_n y^n$, the series $\Sigma c_n A_n$ will converge when $C > r$. Suppose now that A_n depends upon x and put $A_n = G_n(x)$. It follows then from my theorem that $\Sigma c_n G_n(x)$ will *always* converge when $C > r$. But this is what was to be proved.

At the time of the publication of my work I was not aware of Poincaré's article, and I therefore failed to point out the relation of the two memoirs.

$$(n^2+1)G_{n+2}(x) - 2n^2xG_{n+1}(x) + (n^2+x^2)G_n(x) = 0.$$

For $n=\infty$ the limiting form of this equation is

$$G_{n+2}(x) - 2xG_{n+1}(x) + G_n(x) = 0,$$

or the same as the limiting form for the zonal harmonic. The auxiliary equation is

$$z^2 - 2xz + 1 = 0,$$

of which the roots are

$$z = x \pm \sqrt{x^2-1}.$$

The curves $|x \pm \sqrt{x^2-1}| = C$ are easily seen to be ellipses having the foci ± 1. Hence if R is the radius of convergence of $\Sigma c_n y^n$, the region of convergence of (2) is the interior of an ellipse,

$$|x \pm \sqrt{x^2-1}| = R.$$

Poincaré also examines such exceptional cases as that which is specified by relation (3), which has no proper limiting form. But upon this work we can not longer dwell. I wish, however, to emphasize its fundamental character, inasmuch as many previous, and even subsequent conclusions concerning the convergence of series of the form (2) are comprised in *Poincaré's* result.

Somewhat earlier in the lecture I set forth the arbitrary character of the function which could be represented by series of polynomials and rational fractions. We have seen also how this arbitrary element was entirely eradicated by confining ourselves to polynomials which obey a linear law of recurrence. In the remainder of this lecture I wish to develop the consequences of restricting a series of rational fractions in the manner supposed by *Borel* in his thesis * and its recent continuation in the *Acta Mathematica*.† *Borel* seeks to so restrict a series of rational fractions, $\Sigma P_n(x)/R_n(x)$, as to ensure a connection between the position of the poles of its separate terms and the position of the singular points of the function which the series collectively represents. On this account he assigns

* *Ann. de l'Ec. Nor.*, ser. 3, vol. 12 (1895), p. 1.

† Vol. 24 (1900), p. 309.

an upper limit to the degrees of $P_n(x)$ and $R_n(x)$. But this is not enough, and he proceeds therefore to limit the magnitude of the coefficients in the numerators. On the other hand, he allows any distribution whatsoever for the roots of the denominators, thus leaving himself at liberty to vary greatly the nature of the function represented.

In his thesis he develops the case

$$\phi(z) = \sum_{n=1}^{\infty} \frac{A_n}{(z-a_n)^{m_n}} \qquad (m_n \leqq m) \tag{6}$$

which had been previously considered by *Poincaré* * and *Goursat.*† To avoid semi-convergent series or, in other words, functions, of which the character depends not merely upon the position of the poles a_n and the values of A_n but also upon the order of summation, the condition is imposed that ΣA_n shall be absolutely convergent. Then if there is any area of the z plane which contains no poles, the series (6) must converge within this region. Since furthermore it is uniformly convergent in any interior sub-region, it defines an analytic function within the area. There may be several such areas separated by lines or regions in which the poles are everywhere dense. This is precisely the case to be considered now.

To simplify matters, let us suppose that the poles are everywhere dense along certain closed curves of ordinary character, but nowhere inside the curves. *Poincaré* and *Goursat* show that each curve is a natural boundary for the analytic function $\phi(z)$ defined by (6) in its interior. *Borel's* proof is as follows. Denote the component of (6) which corresponds to a_n by

$$\phi_1(z) = \frac{B_m}{(z-a_n)^m} + \frac{B_{m-1}}{(z-a_n)^{m-1}} + \cdots + \frac{B_1}{z-a_n}$$

and the remaining part by

* *Acta Societatis Fennicæ,* vol. 12 (1883), p. 341, and *Amer. Journ. of Math.* vol. 14 (1892), p. 201.

† *Compt. Rend.*, vol. 94 (1882), p. 715.

$$\phi_2(z) = \sum_{i=1}^{r} \frac{A'_i}{(z-\beta_i)^{m_i}} + \sum_{i=r+1}^{\infty} \frac{A''_i}{(z-\beta_i)^{m_i}}.$$

It is evident that if a_n lies within any one of the curves considered, a_n is a pole of $\phi(z)$. Now when these interior poles condense in infinite number in the vicinity of any point of the curve, it must, of course, be a singularity of $\phi(z)$. Consider next any one of the points a_n which lies upon the boundary but is not a point of condensation of the interior poles, and let z approach this point along the normal. Describe a circle upon the line $z - a_n$ as diameter. If z is sufficiently near to a_n, the circle will exclude every one of the points a_i, excepting a_n which lies upon its boundary. Since also ΣA_n is absolutely convergent, by increasing r the second component of $\phi_2(z)$ may be made less in absolute value than $\epsilon/|z-a_n|^m$, in which ϵ is an arbitrarily small prescribed quantity. If, then, H denotes the maximum of the first component of $\phi_2(z)$ as z now moves up to a_n, we have

$$|\phi_2(z)| < H + \frac{\epsilon}{|z-a_n|^m}.$$

Consequently,

$$\lim_{z=a_n} \phi(z)\cdot(z-a_n)^m = \lim \phi_1(z)\cdot(z-a_n)^m + \lim \phi_2(z)\cdot(z-a_n)^m = B_m.$$

This shows that $|\phi(z)|$ increases indefinitely when z approaches any pole a_n of the mth order along a normal, and removes the possibility that the poles, because they are infinitely thick upon the curve, may so neutralize one another that the function can be carried analytically across the curve at a_n. As, moreover, we suppose the points a_n of order m to be everywhere dense upon the curve, it must be a natural boundary.

It is apparent now that the expression (6) continues the initial function $\phi(z)$ across a natural boundary into other regions where it defines in similar manner other analytic functions with natural boundaries. But, it may be asked, is there any proper sense in which these analytic functions may be regarded as a continuation of one another? Just here *Borel* steps in and, after imposing further conditions, shows that when the function defined by (6) within some one of the curves is zero, the functions defined within the other curves must also vanish.* Take $m = 1$, so that

$$(7) \qquad \phi(z) = \Sigma \frac{A_n}{z - a_n}.$$

By a linear transformation

$$z' = \frac{az + b}{cz + d}$$

any interior point of one curve may be taken as the origin and any interior point of a second curve may be transformed simultaneously into the point at infinity without changing the character of the series to be investigated. Now at the origin the successive coefficients in the expansion of $\phi(z)$ into a Taylor's series are the negative of

$$(8) \qquad \Sigma \frac{A_n}{a_n}, \quad \Sigma \frac{A_n}{a_n^2}, \quad \Sigma \frac{A_n}{a_n^3}, \cdots$$

while those in the expansion for $z = \infty$ are

$$(9) \qquad \Sigma A_n, \quad \Sigma A_n a_n, \quad \Sigma A_n a_n^2, \cdots.$$

Borel proves that when

$$\lim_{n=\infty} \sqrt[n]{A_n} = 0,$$

the coefficients (9) must vanish if those given in (8) do. Any one of the analytic functions under discussion is therefore completely determined by any other, the expression (7) being the intermediary by which we pass from one to the other.

So far as yet appears, this method of continuing an analytic function across a natural boundary is of very limited applicability. Its significance has been made clearer by *Borel's* later memoir in the *Acta Mathematica*. Here the rational fractions are of a less highly specialized character, but the essential nature of the investigation can still be exhibited without abandoning the expression (6). Let $|A_n| < u_n^{m+1}$, where u_n denotes the nth term of a convergent series

* Cf. pp. 32–33 of his thesis or pp. 94–98 of his *Théorie des fonctions.*

of positive numbers. We shall suppose that the poles of the terms of (6) are everywhere dense over a large portion of the plane, leaving, however, at least one area free from poles, so that there shall be an analytic function to continue, though even this is not necessary. *Borel* proves that parallel to any assigned direction there will be an infinity of straight lines, everywhere dense throughout the plane, along which the series (6) will converge absolutely and uniformly. The function defined along these lines is therefore a continuous one.

The proof of this result is short and simple. Describe about the poles a_n as centers circles which have successively the radii $u_n (n = 1, 2, \cdots)$. If there is any point which lies outside all of these circles, the series (6) must there converge, since at such a point the absolute value of the nth term is

$$\left|\frac{A_n}{(z - a_n)^{n_m}}\right| < \frac{u_n^{m+1}}{u_n^m} = u_n, \qquad (n > N),$$

that is, less than the nth term of a convergent series of positive numbers. But are there points outside of all the circles? To settle this question, take any straight line perpendicular to the assigned direction and project orthogonally all the circles upon the line. The total sum of all the projections, $2\Sigma u_n$, will be convergent. Moreover, by cutting off a sufficient number of terms at the beginning of (6), the sum of the projections may be made less than any assigned segment ab of the line. Let N terms be cut off for this purpose. Take any point c of the segment which does not lie upon the projection of any circle nor coincide with the projection of one of the first N poles of (6). At c erect a perpendicular to ab. This will be a line parallel to the assigned direction which throughout its entire extent lies without all the circles, excepting possibly the first N. Hence the series (6) will converge absolutely and uniformly along the line, even though the line lie infinitesimally close to some set of poles in the system. Lastly, because ab was an interval of arbitrary length, these lines of convergence must be everywhere dense throughout the plane, obviously forming a non-enumerable aggregate.

Since the series is uniformly convergent, it can be integrated term by term. Clearly also the numerators A_i in (6) can be so conditioned that the term-by-term derivative of (6) shall be uniformly convergent. Then the derivative of $\phi(z)$ is coincident with the derivative of the series. It is even possible to so choose the A_i that the series will be unlimitedly differentiable.

I may add that in any region of the plane there will be an infinite or, more specifically, a non-enumerable set of points, through each of which passes an infinite number of lines of convergence. If a closed curve is given it will be possible to approximate as closely as desired to this curve by a rectilinear polygon, along whose entire length the series converges and defines a continuous function. Integration around such a polygon gives for the value of the integral the product of $2i\pi$ into the sum of the residues of those fractions whose poles lie in the interior of the polygon. Finally, if we take for axes of x and y two perpendicular lines of continuity of $\phi(z)$, all the lines of uniform continuity which meet at their intersection will give a common value for $\phi'(z)$, and the real and imaginary parts of $\phi(z)$ will satisfy *Laplace's* equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus we have in $\phi(z)$ a species of quasi-monogenic function. One question *Borel* has as yet found himself unable to resolve. If $\phi(z) = 0$ along a finite portion of any line, will the series in consequence vanish identically? If this question be answered in the affirmative, the analogy with an ordinary analytic function will be still more complete.

Let us now return to the case in which two or more functions with natural boundaries are defined by (7). The lines of continuity just described form an infinitely thick mesh-work along which $\phi(z)$ can be carried continuously from the one analytic function into the others. Suppose again that the origin is not a point of condensation of the poles a_n so that $\phi(z)$ can be expanded at the origin into a Maclaurin's series $\Sigma c_i z^i$. Now if a ray is drawn from the origin through the pole a_n and the portion of the ray between a_n and ∞ is retained as a cut, the mth term of (7) can be expanded into a series of polynomials

$$-\frac{A_m}{a_m}\sum_{n=1}^{\infty} G_n\left(\frac{z}{a_m}\right)$$

which converges over the plane so cut. The series (7) can therefore be resolved into a double series

$$\sum_m \sum_n \frac{-A_m}{a_m} G_n\left(\frac{z}{a_m}\right),$$

and this expression will be valid on an infinity of rays from the origin which do not pass through any of the poles. Since, moreover, the poles are an enumerable set of points, these rays will be infinitely dense between any two arguments which may be taken. By further conditioning the A_n, *Borel* is able to rearrange the terms of the double series so as to form a series of polynomials $\sum_n Q_n(z)$, in which

$$Q_n(z) = -\sum_{m=1}^{\infty}\frac{A_m}{a_m} G_n\left(\frac{z}{a_m}\right),$$

and in this way he obtains a series of polynomials which is convergent on a dense set of rays through the origin.

It also appears that the polynomial series $\Sigma\, Q_n(z)$ can be formed directly from $\Sigma c_i z^i$ without the intervention of (7). When, therefore a Maclaurin's series is given which corresponds to such an expression (7) as is now under discussion, the continuation of the function can be made along the above set of rays. Now the rays cut any curve upon which either (7) or $\Sigma\, Q_n(z)$ defines a continuous function in a set of points everywhere dense. The value of the function along the entire curve therefore depends only upon the coefficients c_i; *i. e.*, upon the value of the function and its derivatives at the origin. It is shown, moreover, that any point of the plane which is not a point of condensation of the poles a_n may be converted by transformation of axes into such an origin. Finally, *Borel* gives a case in which the poles may be everywhere dense over the entire plane, so that the function defined by (7) is nowhere analytic, and yet its value is determined along the lines of continuity by the value of the function and its derivatives at the origin. Here then is a class of non-analytic functions sharing a most fundamental property in common with the analytic functions! Is it not then possible, as *Borel* surmises, that there is a wider theory of functions, similar in its outlines to the theory of analytic functions and embracing this as a special case? If so, the conceptions of *Weierstrass* and of *Meray* are capable of generalization.

Part II. On Algebraic Continued Fractions.

Lecture 5. *Padé's Table of Approximants and its Applications.*

Both historically and prospectively one of the most suggestive and important methods of investigating divergent power series is by the instrumentality of algebraic continued fractions. It is for this reason that I have ventured to combine in a single course of lectures two subjects apparently so unrelated as divergent series and continued fractions. I shall not, however, confine myself to the consideration of the latter subject solely with reference to the theory of divergent series. It is rather my purpose to give some account of the present status of the theory of algebraic continued fractions. At the close of the next lecture a bibliography of memoirs connected with the subject is appended, to which reference is made throughout this lecture and the next by means of numbers enclosed in square brackets.

By the term *algebraic* continued fraction is understood, in distinction from a continued fraction with numerical elements, one in which the elements — *i. e.*, the partial numerators and denominators — are functions of a single variable x or of several variables [16, a, p. 4]. Although the term algebraic does not seem to me to be fortunately chosen, I shall nevertheless accept it and use it to indicate the class of continued fractions which it is proposed to consider here.

The first foundations of a theory of continued fractions were laid by *Euler*, who early employed them [1, a] to derive from a given power series

$$k_0 + k_1x + k_2x^2 + \cdots \qquad (k_n \neq 0)$$

a continued fraction of the form

$$(1) \qquad \frac{1}{b_0} + \frac{a_1x}{b_1x + d_1} + \frac{a_2x}{b_2x + d_2} + \cdots.$$

A second form, also introduced by *Euler* * [46, a] is the more familiar one

$$(2) \qquad \frac{a_0}{1} + \frac{a_1x}{1} + \frac{a_2x}{1} + \frac{a_3x}{1} + \cdots,$$

which was later used by *Gauss* [34] in his celebrated continued fraction for $F(\alpha, \beta, \gamma, x)/F(\alpha, \beta + 1, \gamma + 1, x)$. From this time on still other forms were discovered so that it became impossible to speak of a unique development of a function into a continued fraction. Among these forms may be especially mentioned the continued fraction

$$(3') \qquad \frac{1}{a_1x + b_1} + \frac{1}{a_2x + b_2} + \frac{1}{a_3x + b_3} + \cdots,$$

used by *Heine, Tchebychef,* and others in approximating to series in descending powers of x. By the substitution of $1/x$ for x and a simple reduction this can be transformed, after the omission of a factor x, into

$$(3) \qquad \frac{1}{a_1 + b_1x} + \frac{x^2}{a_2 + b_2x} + \frac{x^2}{a_3x + b_3} + \cdots.$$

The reason for this variety of form and for the occurrence, in

* Padé in his thesis (p. 38) traces it back to Lambert [2, a] and Lagrange, but Euler's use is earlier still.

particular, of the three types just given is discussed by *Padé* in his thesis [16, *a*]. As this thesis is the foundation for a systematic study of continued fractions, it will be necessary to give a recapitulation of its chief results.

Let

$$S(x) = c_0 + c_1 x + c_2 x^2 + \cdots \qquad (c_0 = 1) \tag{4}$$

be any given power series, whether convergent or divergent. If $N_p(x)/D_q(x)$ denotes an arbitrary rational fraction in which the numerator and denominator are of the pth and qth degrees respectively, there will be $p + q + 1$ parameters which can be made to satisfy an equal number of conditions. Let them be so determined that the expansion of N_p/D_q in ascending powers of x shall agree with (4) for as great a number of terms as possible. In general, we can equate to zero the first $p + q + 1$ coefficients of the expansion of $D_q S(x) - N_p$ in ascending powers of x, and no more. Hence, unless N_p and D_q have a common divisor, the series for N_p/D_q agrees with (4) for an equal number of terms, and the approximation is said to be of the $(p + q + 1)$th order. In exceptional cases the order of the approximation may be either greater or less. *Padé* examines these exceptional cases and proves strictly that among all the rational fractions in which the degrees of numerator and denominator do not exceed p and q respectively, there is, taken in its lowest terms, one and only one, the expansion of which in a series will agree with (4) for a greater number of terms than any other. Such a rational fraction I shall term an *approximant* of the given series.

The existence of approximants was, of course, well known before *Padé*, but no systematic examination of them had been made except by *Frobenius* [13], who determined the important relations which normally exist between them. *Padé* goes further, and arranges the approximants, expressed each in its lowest terms, into a table of double entry :

	$p=0$	$p=1$	$p=2$	$\dots$
$q=0$	$\frac{N_{00}}{D_{00}} = c_0$	$\frac{N_{10}}{D_{10}} = c_0 + c_1x$	$\frac{N_{20}}{D_{20}} = c_0 + c_1x + c_2x^2$	$\dots$
$q=1$	$\frac{N_{01}}{D_{01}}$	$\frac{N_{11}}{D_{11}}$	$\frac{N_{21}}{D_{21}}$	$\dots$
$q=2$	$\frac{N_{02}}{D_{02}}$	$\frac{N_{12}}{D_{12}}$	$\frac{N_{22}}{D_{22}}$	$\dots$
$q=3$	. . . .	. . . .	. . . . .	.

When the order of approximation of a rational fraction, taken in its lowest terms, is exactly equal to the sum of the degrees of numerator and denominator, increased by 1, the fraction will be found once and only once in the table. If, conversely, a fraction N_{pq}/D_{pq} occurs but once in the table, the numerator and denominator are of degree p and q respectively, and the order of the approximation which the fraction affords is exactly $p+q+1$. The approximant is then said by *Padé* to be normal. We shall also call the table *normal* when it consists only of normal fractions, or, in other words, when no approximant occurs more than once in the table.

Obviously all approximants which lie upon a line perpendicular to the principal diagonal of the table correspond to the same value of $p+q+1$. Hence in a normal table they approximate to (4) in equal degree, and accordingly may be said to be *equally advanced* in the table. If $p+q+1$ increases in passing from one fraction to another, the latter is the more advanced.

Two approximants will be called *contiguous* if the squares of the table in which they are contained have either an edge or a vertex in common.

Consider now a normal table, and take any succession of approximants, beginning with one upon the border of the table and passing always from one approximant to another which is contiguous to it but more advanced. *Padé* shows that any such sequence of approximants makes a continued fraction of which the approximants are the successive convergents. * Thus a countless manifold of continued fractions can be formed, any one of which through its convergents gives the initial series to any required number of terms and hence defines the series and table uniquely. In all of *Padé's* continued fractions the partial numerators are monomials in x.

The continued fraction is called regular when its partial numerators are all of the same degree and likewise its denominators, certain specified irregularities being admitted in the first one or two partial fractions. These irregularities disappear when the continued fraction, as is most usual, commences with the corner element of the table. (Cf. the continued fractions (2) and (3).)

In a normal table a regular continued fraction can be obtained in any one of three ways. If we take for the convergents the approximants which fill a horizontal or vertical line, a continued fraction is obtained which — except for the irregularity permitted at the outset — is of the form (1) given above. If the approximants lie upon the principal diagonal or any parallel line, the continued fraction is of type (3). Lastly, if the convergents lie upon a stair-like line, proceeding alternately one term horizontally to the right and one term vertically downward, the continued fraction is of the familiar form (2).

When a table is not normal, the approximants which are identical with one another are shown by *Padé* to fill always a square, the edges of which are parallel to the borders of the table. When the square contains $(n + 1)^2$ elements, the irregularity may be said to be of the nth order. The vertical, horizontal, diagonal and stair-like lines give regular continued fractions as before, unless they cut into one or more of these square blocks of equal approximants. When this happens, certain irregularities appear in the continued fraction which give rise to various difficulties in the consideration of matters of convergence and other questions.

On this account it is natural to inquire first whether the continued fraction has or has not a normal character. If it has, the

---

* This is also tacitly implied in the relations given by Frobenius [13, p. 5].

existence of the three regular types of continued fractions is assured. The necessary and sufficient condition that the table shall be normal is that no one of the determinants

$$c_{\alpha\beta} = \begin{vmatrix} c_{\alpha-\beta+1} & c_{\alpha-\beta+2} & \cdots & c_{\alpha} \\ c_{\alpha-\beta+2} & c_{\alpha-\beta+3} & \cdots & c_{\alpha+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{\alpha} & c_{\alpha+1} & \cdots & c_{\alpha+\beta-1} \end{vmatrix} \qquad \begin{matrix}(\alpha,\ \beta \geqq 0; \\ \quad c_i = 0 \text{ if } i < 0)\end{matrix}$$

shall vanish [16, a, p. 35]. It will be noticed that the determinants are of the same sort as those which play so conspicuous a rôle in *Hadamard's* discussion of series representing functions with polar singularities.

So far as I am aware, the normal character of the table has been established as yet only in the following cases: (1) for the exponential series [37] and for $(1 + x)^m$ when m is not an integer [35, d],† by *Padé*; and (2) for the series of *Stieltjes*, by myself [45].

The construction of *Padé's* table leads at once to a number of new and important questions. The numerators and the denominators of the approximants constitute groups of polynomials which it is only natural to expect will be characterized by common or kindred properties. The table then affords a suitable basis for the classification of polynomials. Thus, for example, the polynomials of

† At least half of the table for $F(\alpha, 1, \gamma, x)$ has a normal character. This was proved incidentally in my thesis [76] by showing that the remainders corresponding to approximants on or above the diagonal of the table were all distinct. The method of conformal representation was there employed, but the same fact can also be demonstrated very simply by means of Gauss' *relationes inter contiguas* (formulas (19) and (20) of [34]). The approximants in the other half of my table (Cf. [76], p. 44) were constructed on different principles from Padé's, the approximation being made simultaneously with reference to *two* points, $x = 0$ and $x = \infty$, but the resulting continued fractions were of the same form as Padé's. It is noteworthy that the *relationes inter contiguas* lead to such a table rather than to the one of Padé's construction.

In the case of $F(-m, 1, 1, -x) \equiv (1 + x)^m$ the half of Padé's table below the diagonal is also normal, since the reciprocal of the approximants in the lower half are the approximants in the upper half of the table for

$$F(m, 1, 1, -x) = (1 + x)^{-m}.$$

The normal character of the table for e^x then follows since $e^x = \lim_{g=\infty} F(g, 1, 1, x/g)$.

Legendre and similar polynomials are obtained from the series for $\log(1-x)/(1+x)$, while the numerators and denominators of the approximants for $(1+x)^m$ are the hypergeometric polynomials $F(-\mu, -\nu \pm m, -\mu+\nu, -x)$, in which μ and ν are integers, or the so-called polynomials of *Jacobi* [65]. In these, as in numerous other cases, the denominators of the convergents and the remainder-functions,* formed by multiplying each denominator into the corresponding remainder, are solutions of homogeneous linear differential equations of the 2nd order which have a common group, and the relations of recurrence between three successive denominators or remainder-functions are the *relationes inter contiguas* of *Gauss* and *Riemann*. (See in particular, [75, *d*] and [76].) The further study of such groups of polynomials will probably bring to light new and important properties. The position of the roots of the denominators should especially be ascertained, because the distribution of these roots has an intimate connection with the form of the region of convergence of the continued fraction and oftentimes also with the position and character of the function which the continued fraction defines.

Probably the most fundamental question concerning *Padé's* table is that of the convergence of the various classes of continued fractions or lines of approximants. The first investigation of the convergence of an algebraic continued fraction was made by *Riemann* [18] in 1863, followed by *Thomé* [19] a few years later.† Both writers investigated the continued fraction of *Gauss* by rather painful methods, not based absolutely upon the algorithm of the continued fraction but upon extraneous considerations. This is not surprising, for there were at that time no general criteria for the convergence of continued fractions with *complex* elements, and even now the number is astonishingly small.

---

* In at least half of the table. See the preceding footnote.

† As Riemann's work appeared posthumously, Thomé's has the priority of publication (1866) but was itself preceded by Worpitzky's dissertation, to which reference is made in a subsequent footnote.

The two principal criteria for convergence correspond to the familiar tests for the convergence of a real continued fraction

$$(5) \qquad \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} + \cdots,$$

in which either (1) all the elements are positive or (2) the partial denominators λ_i are positive and the partial numerators μ_i are negative. The latter class of real continued fractions is known to converge if $\lambda_i \geqq 1 - \mu_i$. *Pringsheim* [29] has shown that when the elements are complex, the condition $|\lambda_i| \geqq 1 + |\mu_i|$ is still sufficient for convergence. If, furthermore, the continued fraction has the customary normal form in which $\mu_n = 1$, the condition may be replaced by the less restrictive one [29, p. 320],

$$\left|\frac{1}{\lambda_{2n-1}}\right| + \left|\frac{1}{\lambda_{2n}}\right| \leqq 1.$$

The necessary and sufficient condition for the convergence of the first class of real continued fractions can be most easily expressed after it has been reduced to the form

$$\frac{1}{\lambda_1'} + \frac{1}{\lambda_2'} + \frac{1}{\lambda_3'} + \cdots \qquad (\lambda_n' > 0).$$

If then $\Sigma\lambda_n'$ is divergent, the continued fraction converges, while it diverges if $\Sigma\lambda_n'$ is convergent.* But in the latter case limits exist for the even and the odd convergents when considered separately. This result is included in the following theorem which I gave in the *Transactions* of 1901 for continued fractions with complex elements [31]: If in

$$\frac{1}{\alpha_1 + i\beta_1} + \frac{1}{\alpha_2 + i\beta_2} + \frac{1}{\alpha_3 + i\beta_3} + \cdots$$

the elements α_i have all the same sign and the β_i are alternately positive and negative,† the continued fraction will converge if $\Sigma|\alpha_n + i\beta_n|$ is divergent; on the other hand, if $\Sigma|\alpha_n + i\beta_n|$ is

*Seidel, *Habilitationsschrift*, 1846, and Stern, *Journ. für Math.*, vol. 37 (1848), p. 269.

† Zero values are permissible for either α_i or β_i.

convergent and *either* the α_i *or* the β_i fulfill the condition just stated concerning their signs, the even and the odd convergents have separate limits.

The most general criterion for the convergence of

$$\frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} + \cdots$$

(b_i real or complex) seems to be the one which I gave in October, 1901 [32, *b*, § 5].

Two remarks of a general nature concerning the convergence of algebraic continued fractions may be of interest. In the consideration of numerical continued fractions a difficulty frequently encountered is that the removal of a finite number of partial fractions μ_i/λ_i at the beginning of (5) may affect its convergence or divergence. The convergence is therefore not determined solely by the ultimate character of the continued fraction, as is true of a series. *Pringsheim* [29] has proposed to call the convergence *unconditional* when it is not destroyed by the removal of the first n partial fractions of (5). The difficulties due to conditional convergence usually disappear from consideration in treating algebraic continued fractions. For let N_n/D_n now denote the nth convergent. If after the removal of the first n partial fractions the continued fraction converges uniformly in a given region and accordingly represents a function $F(z)$ which is holomorphic within the region, then after the restoration of the initial terms the continued fraction will define the function

$$\frac{N_n + F(z)N_{n-1}}{D_n + F(z)D_{n-1}}, \tag{6}$$

which must be either holomorphic or meromorphic within the given region [32, *a* or *c*]. An exception occurs only if the denominator of (6) vanishes identically in the region. This is impossible for the second and third types of continued fractions, since the development of a rational fraction $-D_n/D_{n-1}$ in either type (2) or (3) consists of a finite number of terms, whereas the development of $F(z)$, by hypothesis, continues indefinitely.

The second remark relating to convergence is that its discussion for a continued fraction is usually reduced to the corresponding question for an infinite series. The succession of convergents

$$\frac{N_n}{D_n}, \quad \frac{N_{n+1}}{D_{n+1}}, \quad \frac{N_{n+2}}{D_{n+2}}, \quad \cdots$$

is, in fact, obviously equivalent to the series

$$\frac{N_n}{D_n} + \left(\frac{N_{n+1}}{D_{n+1}} - \frac{N_n}{D_n}\right) + \left(\frac{N_{n+2}}{D_{n+2}} - \frac{N_{n+1}}{D_{n+1}}\right) + \cdots.$$

But the latter by means of the familiar relations connecting the denominators or the numerators of three consecutive convergents may be reduced to the form:

$$(7) \qquad \frac{N_n}{D_n} + (-1)^n \mu_1 \mu_2 \cdots \mu_n \left(\frac{\mu_{n+1}}{D_n D_{n+1}} - \frac{\mu_{n+1}\mu_{n+2}}{D_{n+1}D_{n+2}} + \frac{\mu_{n+1}\mu_{n+2}\mu_{n+3}}{D_{n+2}D_{n+3}} - \cdots\right).$$

We turn now from these general considerations to the questions of convergence connected with *Padé's* table. Under what conditions will the various lines of approximants converge; in particular, the three standard types of continued fractions obtained by following (1) the horizontal or vertical lines, (2) the stair-like lines, and (3) the diagonal lines? When they converge simultaneously, have they a common limit? If not, what are the mutual relations between the functions which they define? What is the form of the region of convergence?

These and other questions press upon us, and are of great interest. A complete investigation has been made only for the exponential series. *Padé* [37, a] finds that when p/q for any succession of approximants N_{pq}/D_{pq} converges to a value ω, the approximants converge toward the generating function e^x for all values of x. Furthermore, the numerators and denominators separately converge, the former to the limit $e^{\omega x/\omega+1}$, the latter to $e^{-x/\omega+1}$. This smooth result is not, however, a typical one, not even for entire functions. It is due at least in part to the fact that e^x is an entire function without zeros. This will be apparent after an examination has been made of the vertical and horizontal lines of *Padé's* table, which we now proceed to consider.

It is obvious that the first $p + q + 1$ terms of the given series (4) determine an equal number of terms of the series for its reciprocal. If, therefore, in the table each approximant is replaced by its reciprocal and the rows and columns are then interchanged, we shall obtain the table for the reciprocal series. The problems presented by the horizontal and vertical lines of the table are consequently of essentially the same character, and our attention may be confined henceforth to the horizontal lines alone.

By the interchange just described the zeros and poles of (4) become the poles and zeros respectively of the reciprocal function. In the case of the exponential function the reciprocal series has the same character as the initial series, each defining an entire function without zeros, and the simultaneous convergence of rows and columns for all values of x was therefore to be expected; but in general this does not hold.

In investigating the convergence of the horizontal lines the first case to be considered is naturally that of a function having a number of poles and no other singularities within a prescribed distance of the origin. It is just this case that *Montessus* [33, *a*] has examined very recently. Some of you may recall that four years ago in the Cambridge colloquium Professor *Osgood* * took *Hadamard's* thesis † as the basis of one of his lectures. This notable thesis is devoted chiefly to series defining functions with polar singularities. *Montessus* builds upon this thesis and applies it to a table possessing a normal character. Although his proof is subject to this limitation, his conclusion is nevertheless valid when the table is not normal, as I shall show in some subsequent paper.

The first horizontal row of the table scarcely needs consideration, for it consists of the polynomials obtained by taking successively 1, 2, 3, $\cdots$ terms of the series. Consequently the continued fraction obtained from the first row,

---

* *Bull. of the Amer. Math. Soc.*, vol. 5, pp. 74–78.

† *Journ. de Math.*, ser. 4, vol. 8 (1892).

$$\frac{a_0}{1} - \frac{a_1 x}{a_1 x + a_0} - \frac{a_0 a_2 x}{a_2 x + a_1} - \frac{a_1 a_3 x}{a_3 x + a_2} - \cdots,$$

is identical with the series, and its region of convergence is a circle.

Let R_1 be the radius of this circle and q_1 the number of poles of (4) which lie upon its circumference. Suppose also that the next group of poles, q_2 in number, lie upon a circle of radius R_2, having its center in the origin; that q_3 poles lie upon the next circle (R_3); and so on indefinitely or until a circle is reached which contains a non-polar singularity. *Hadamard* (*l. c.*, § 18) has proved that the denominators D_{pq} of the approximants of the $(q_1 + 1)$th row, of the $(q_1 + q_2 + 1)$th row, and so on, approach a limiting form as we advance in the row, and that the limiting polynomials give the positions of the first $q_1, q_1 + q_2, \cdots$ poles respectively. Thus if, for example,

$$D_{p,\,q_1+1} = 1 + B^{(1)}_{pq_1} x + B^{(2)}_{pq_1} x^2 + \cdots + B^{(q_1)}_{pq_1} x^{q_1}$$

and

$$\lim_{p=\infty} B^{(i)}_{pq_1} = B_i,$$

the first group of poles are the roots of the polynomial $1 + B_1 x + \cdots B_{q_1} x^{q_1}$. Using this result of *Hadamard*, *Montessus* shows that in a normal table the approximants of the $(q_1 + 1)$th row converge at every point within the circle (R_2) — excepting, of course, at the q_1 poles — but not without this circle; that the approximants of the $(q_1 + q_2 + 1)$th row converge similarly within the circle (R_3) except at the included $q_1 + q_2$ poles; and so on.

In proving this *Montessus* makes use of an idea advanced in *Padé's* thesis ([16, *a*, p. 51], or [24]) which, though applicable in the present case, is possibly somewhat misleading. In *Padé's* continued fractions the partial numerators μ_i are monomials in x. This is due to the fact that there is a steady increase in the order of the approximation afforded by the successive convergents at $x = 0$. Consider now the series (7), and let T denote the region or set of points in the x-plane for which $|D_n|$, from and after some value of n, has both an upper and a lower limit. Then in T the continued fraction will converge or diverge simultaneously with the power series,

$$\mu_{n+1} - \mu_{n+1}\mu_{n+2} + \mu_{n+1}\mu_{n+2}\mu_{n+3} \cdots . \tag{8}$$

Call C the circle of convergence of (8). At all points of T within C the continued fraction converges, and at all exterior points of T it diverges. On this account *Padé* proposes to call C the "circle of convergence" of the continued fraction. In the case which we have just been discussing this concept is applicable because of the existence of limiting forms for the denominators of the rows considered. The region T comprises the entire finite plane with the exception of the roots of the limiting form, and the circle C is successively identical with (R_2), (R_3), ... Thus, as we pass down the rows of the table, we obtain continued fractions having an increasing region of convergence.

In introducing the term circle of convergence for a continued fraction *Padé* ignores all points not included in T. Call the excluded point-set T'. If $|D_n|$ increases indefinitely with increasing n over the whole or a part of T' the series (7) may converge, and this may happen even though (8) is a divergent series.* The term circle of convergence is therefore an inappropriate one, although the considerations upon which it is based are useful.

Nothing more of account seems to be known concerning the the convergence of the horizontal and vertical lines.† The more common and important continued fractions are obtained from diagonal and stair-like paths through the table. In many familiar continued fractions of the second type,

$$\frac{a_0}{1} + \frac{a_1 x}{1} + \frac{a_2 x}{1} + \frac{a_3 x}{1} + \cdots, \tag{2}$$

---

* The coefficients in the continued fraction of Stieltjes (discussed later in the lecture) can be easily so determined as to give a case of this sort, the region of convergence of (7) being the entire plane with the exception of the negative half of the real axis. We suppose, with Padé that the absolute term of D_n is taken equal to 1.

† It is perhaps worth noting that the coefficients in the first type of continued fractions can not be selected arbitrarily if it is to be connected with such a table as Padé constructs. In the other two types the coefficients are entirely arbitrary.

a_n with increasing n approaches a limit, as for instance in the continued fraction of *Gauss* where $\lim a_n = -\frac{1}{4}$. The significance of the existence of such a limit I first pointed out for a comprehensive class of cases in 1901 [32, *a*], and since then I have shown by simpler methods [32, *c*] that the result is perfectly general. Let $\lim a_n = k$. Then the continued fraction converges, save at isolated points, over the entire plane of x with the exception of the whole or a part of a cut drawn from $x = -1/4k$ to $x = \infty$ in a direction which is a continuation of the vector from $x = 0$ to $x = -1/4k$. Within the plane thus cut the limit of the continued fraction is holomorphic except at the isolated points which (if they exist) are poles. When there is no limit for a_n but only an upper limit U for its modulus, the continued fraction (see [32, *b*]) is meromorphic or holomorphic at least within a circle of radius $1/4U$ having its center in the origin.* A special case is that in which $\lim a_n = 0$. The limit of the continued fraction is then a function which is holomorphic or meromorphic over the entire plane. A comparison of this last result with that of *Montessus* shows that a much greater region of convergence has now been obtained. This is doubtless, in general, a reason for preferring the second and third types of continued fractions to the first.

As another illustration of the second type of continued fraction I shall choose the celebrated continued fraction of *Stieltjes* [26, *a*]. In this each coefficient a_n is positive. By putting $x = 1/z$ in (2), the continued fraction, after dropping a factor z, can be thrown into the form

$$(8') \qquad \frac{1}{a_1'z} + \frac{1}{a_2'} + \frac{1}{a_3'z} + \frac{1}{a_4'} + \frac{1}{a_5'z} + \cdots, \qquad (a_n' > 0),$$

which is the form preferred by *Stieltjes*. To every such continued fraction there corresponds a series

* A demonstration of this property within the circle $(1/4U)$ has been previously given in a dissertation by *Worpitzky* [18 bis], which has come to my notice for the first time during the examination of the proof-sheets of these lectures. This dissertation bears the date 1865 and appears to be the earliest published memoir treating of the convergence of algebraic continued fractions.

(9)
$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \cdots$$

for which

(10)
$$A_n \equiv \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-2} \end{vmatrix} > 0,$$

$$B_n \equiv \begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_2 & c_3 & c_4 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n-1} \end{vmatrix} > 0.$$

The correspondence is also a reciprocal one. To every series which fulfills these conditions there corresponds a continued fraction of the above type with positive coefficients. From the conditions (10) it follows that $c_i > 0$ and that $c_n/c_{n-1} > c_{n-1}/c_{n-2}$. If, therefore, the increasing ratio c_n/c_{n-1} has a finite limit, the series is convergent. On the other hand, if it increases without limit, the series is divergent.

In investigating the convergence of the continued fraction the especial skill of *Stieltjes* was shown. From the relation connecting three consecutive denominators (numerators) of the convergents it was shown easily that either set of alternate denominators (numerators) made a Sturm's series, whence it follows that all the roots of the denominators (numerators) lie upon the negative half of the real axis of z. This leads naturally to the conjecture that the region of convergence will be the entire plane of z with the exception of the whole or a part of the negative half axis, and that the functional limit will have no zeros exterior to this half of the axis. First the convergence is examined when z is real and positive. The criterion of *Seidel*, cited previously in this lecture, then applies. If, namely, $\Sigma a_n'$ is divergent, the continued fraction will converge along the positive axis, while if $\Sigma a_n'$ is convergent, the two sets of alternate convergents have limits which are distinct. The conclusion is next extended by *Stieltjes* to the half of the complex plane for which the real part of z is positive.

This brings him to the difficult part of his problem, the extension of the result to the other half-plane but with exclusion of the real axis. Here, particularly, *Stieltjes* [26, a, § 30] shows his ingenuity. He overcomes the difficulty by establishing first a preliminary theorem which is of vital importance for sequences of polynomials or rational fractions. The theorem is as follows. Let $f_1(z), f_2(z), \cdots$ be a sequence of functions which are holomorphic within a given region T, and suppose that $\Sigma_{n=1}^{\infty} f_n(z)$ is uniformly convergent in some part T'' of the interior of T. Then if $f_1(z) + f_2(z) + \cdots + f_n(z)$ has an upper limit *independent of* n in any arbitrary region T' which includes T'' but is contained in the interior of T, the series $\Sigma f_n(z)$ will converge uniformly in T' and therefore has as its limit a function which is holomorphic over the whole interior of T*.

In the application of this theorem *Stieltjes* decomposes each convergent $N_n(z)/D_n(z)$ into partial fractions,

$$\frac{M_1}{z+a_1} + \frac{M_2}{z+a_2} + \cdots + \frac{M_r}{z+a_r}$$

in which

$$M_i > 0, \qquad a_i \geqq 0, \qquad \sum_{i=1}^{r} M_i = c_0.$$

From this it follows that N_n/D_n has an upper limit independent of n in any closed region of the plane which does not contain a point of the negative half-axis. If now in either the sequence of the odd convergents or of the even convergents we denote the nth term of the sequence by N_n/D_n and place

$$f_1(z) + f_2(z) + \cdots + f_n(z) = \frac{N_n(z)}{D_n(z)},$$

the series $\Sigma_{n=1}^{\infty} f_n(z)$ converges uniformly in any portion of the plane

* For a further extension of this line of work, see Osgood, *Annals of Math.*, ser. 2, vol. 3 (1901), p. 25.

for which the real part of z is positive. All the conditions of the lemma of *Stieltjes* are now fulfilled, and the region of convergence may be extended over the entire plane with the exception of the negative half-axis.

On account of the uniform character of the convergence the limit of either sequence is holomorphic at every point exterior to the negative half-axis. When $\Sigma a_n'$ is divergent, the two limits coincide and the continued fraction itself is convergent. On the other hand, if $\Sigma a_n'$ is convergent, the two limits are distinct. *Stieltjes* shows also that in the latter case the numerators and the denominators of either sequence converge to holomorphic functions $p(z)$, $q(z)$ of *genre* 0, and the two pairs of functions are connected by the equation

$$q(z)p_1(z) - q_1(z)p(z) = 1,$$

which corresponds to the familiar relation

$$D_{2n}N_{2n-1} - D_{2n-1}N_{2n} = 1.$$

A more direct method [31] of demonstrating the convergence results of *Stieltjes* is by an extension * of the criterion previously cited for the convergence of continued fractions in which the partial fractions $1/(\alpha_n + i\beta_n)$ have an α_n of constant sign and a β_n of alternating sign. The introduction of the lemma of *Stieltjes* is consequently unnecessary, but I wish nevertheless to emphasize its fundamental importance. Other notable results which it will be impossible to reproduce here are also contained in his splendid memoir.

---

* If namely, $\Sigma |\alpha_n + i\beta_n|$ is divergent and the condition concerning the signs either of the α_n or of the β_n is fulfilled, the continued fraction will converge provided $|\alpha_n|/|\beta_n|$ has a lower or an upper limit respectively. Put now $z = w^2$ in (8′) so that it becomes

$$\frac{1}{w}\left(\frac{1}{a_1'w} + \frac{1}{a_2'w} + \frac{1}{a_3'w} + \cdots\right).$$

When $\Sigma a_n'$ is divergent, this falls under the extended criterion if we put $a_n'w = \alpha_n + i\beta_n$, except when z is negative. On the other hand, when $\Sigma a_n'$ is convergent, the criterion applies without extension directly to (8′). In either case the uniform character of the convergence follows with the addition of a few lines.

It is interesting to bring this work of *Stieltjes* into connection with the table of *Padé* [44]. The odd convergents of the continued fraction of *Stieltjes* fill the principal diagonal of *Padé's* table, thus constituting by themselves a continued fraction of the third type, and the even convergents fill the parallel file immediately below, forming a similar continued fraction. The significance of distinct limits for the two sets of convergents is thus made clearer.

The series of *Stieltjes* has perhaps its greatest interest when treated in connection with the theory of divergent series. Although the continued fraction always converges if the series does, the converse is not true. For when the series (9) is divergent, two cases are to be distinguished according as $\Sigma a_n'$ is divergent or convergent. In the former case the continued fraction gives one and only one functional equivalent of the divergent series. *Le Roy* states,* though without proof, that the function furnished is identical with the one obtained from the series by the method of *Borel*, whenever the latter method is applicable also. When $\Sigma a_n'$ is convergent, two different functions are obtained from the continued fraction, the one through the even and the other through the odd convergents. And if there are two such functions which correspond to the series, there must be an infinite number. For if $\phi(x)$ and $\psi(x)$, when expanded formally, give rise to the same divergent series, so also will

$$\frac{\phi(x) + c\,\psi(x)}{1 + c},$$

in which c denotes an arbitrary constant. Special properties, however, attach themselves to the two functions picked out by the continued fraction of *Stieltjes*, upon which we can not linger here.

This result of *Stieltjes* seems to me to be especially significant, since it indicates a division of divergent series into at least two classes, the one class containing the series for which there is properly a single functional equivalent and the other comprising the

* *Loc. cit.*, p. 428.

series which correspond to sets of functions. It is, of course, just possible that this distinction may be due to the nature of the algorithm employed in deriving the functional equivalent of the series, but it is far more probable that the difference is intrinsic and independent of the particular algorithm. If this view be correct, the method of *Borel* which gives a single functional equivalent, is limited in its application to series of the first class.

An extension of the work of *Stieltjes* has been sought in two distinct directions by modification of the conditions imposed upon his series. *Borel* [43] so modifies them as to make the series (when divergent) fulfill the requirement imposed in lecture 2 and permit of manipulation precisely as a convergent series. In the last number of the *Transactions* * [45] I began a study of series which are subject to only one of the two restrictions expressed in the inequalities (10), but was obliged to bring the work to a hurried close to prepare these lectures. In the main, the corresponding continued fractions have the same properties as the continued fraction of *Stieltjes*, but a considerable difference is shown in regard to convergence. Though the roots of the numerators and denominators of the convergents are still real, they are no longer confined to the negative half of the real axis, and may be infinitely thick along the entire extent of the axis. In certain cases the continued fraction converges in the interior of the positive and negative half planes, defining in each an analytic function which has the real axis as a natural boundary. The continued fraction therefore effects the continuation of an analytic function across such a boundary, and gives a natural instance of such a continuation † — natural in distinction from artificial examples set up with the express object of showing the possibility of a unique, non-analytic extension.

Padé [17, *a*] has suggested the foundation of a theory of diver-

---

* July, 1903.

† Earlier instances of a natural continuation are also to be found, as, for example, that afforded by

$$\sum_m \sum_{m'} \frac{1}{(m + m'\omega)^4}$$

across the axis of reals.

gent series upon the continued fractions of his table. The difficulties of carrying out the suggestion are undoubtedly very great and have been pointed out by *Borel.** Not only must the convergence of the principal lines of approximants and the agreement of their limits be investigated, but the combination of two or more divergent series must also be considered. It is not enough to point out, as does *Padé*, that the approximants of given order for any two series, whether divergent or convergent, determine uniquely the approximants of the same or lower order for the sum- and product-series. For practical application of the theory it must be proved also that the function defined by the table corresponding to the new series is, under suitable limitations, the sum or product of the functions defined by the given divergent series. But great as are the difficulties of such an investigation, even for restricted classes of series, the reward will probably be correspondingly great.

So far as it has been yet investigated, the diagonal type of continued fractions seems to have accomplished nearly everthing that can fairly be asked of a sequence of rational fractions. Not only does it afford a convenient and natural algorithm for computing the successive fractions, but in every known instance the region of convergence is practically the maximum for a series of one valued functions. The continued fraction of *Halphen* [21, *a*], so frequently cited as an instance of a continued fraction which diverges though the corresponding series converges, might appear at first sight to be an exception. But this divergence occurs only at special points. In fact, the continued fraction not only converges at the center of the circle of convergence for the series, but, as *Halphen* himself says, continues the function over the entire plane with the exception of certain portions of a line or curve. If then, continued fractions offer such advantages for known series and classes of functions, is it too much to expect that in the future they will throw a powerful searchlight upon the continuation of analytic functions and the theory of divergent series?

---

* *Les Séries divergentes,* p. 60.

LECTURE 6. *The Generalization of the Continued Fraction.*

In the last lecture the algebraic continued fraction was presented under the form of a series of approximants for a given function. An immediate generalization of this conception can be obtained either by increasing the number of points at which an approximation is sought or by requiring a simultaneous approximation to several functions. The latter generalization results at once from an attempt to increase the dimensions of the algorithm or, in other words, the number of terms in the linear relation of recurrence between the successive convergents or approximants. As this generalization is without doubt the more important, I shall make it the chief subject of this lecture. But a few words, at least, should be devoted to the former extension, which is worthy of a more careful and systematic study than it has received.

Denote again by $N_p(x)/D_q(x)$ a rational fraction with arbitrary coefficients. These can, in general, be so determined that its expansion at $x = 0$ shall agree for n_1 successive terms with a given series

$$c_0 + c_1 x + c_2 x^2 + \cdots$$

its expansion at $x = a_1$ for n_2 successive terms with

$$b_0 + b_1(x - a_1) + b_2(x - a_1)^2 + \cdots,$$

at $x = a_2$ for n_3 successive terms with

$$k_0 + k_1(x - a_2) + k_2(x - a_2)^2 + \cdots,$$

and so on, the total number of conditions thus imposed being equal to $p + q + 1$ or the number of parameters in the rational fraction. To each set of values for the n_i and q there corresponds an approximant, and the various approximants can be arranged into a table of multiple entry according to the values of these quantities. Continued fractions, at least in the case of a normal table, can then be obtained by following any path which passes successively from one approximant to another contiguous to it but more advanced in the table. As we proceed along the path, the degree of approximation for each of the points $0, a_1, a_2, \cdots$ must not decrease while at each step it is to increase for at least some one point. The partial numerators of the continued fraction are then either positive integral powers of x, $x-a_1$, $x-a_2$, $\cdots$, or the products of such powers. The degrees of the approximations obtained by stopping the continued fraction with any term can be inferred readily from the degrees of the partial numerators in x, $x-a_1$, $x-a_2$, $\cdots$. The details of the theory have not been worked out.*

The interest of such work can perhaps best be made apparent by referring to the developments for the simplest case in which each n_i is taken equal to 1. The rational fraction N_p/D_q is then completely determined by the requirement that at $p+q+1$ given points $a_1=0$, a_2, a_3, $\cdots$ it shall take an equal number of prescribed values, A_1, A_2, A_3, $\cdots$. If these are the values which a single function assumes at the points, we have the rational fractions which were introduced by *Cauchy* into the theory of interpolation [99, *a*] and which have been quite recently formed into a table and examined by *Padé* [112]. As $p+q+1$ increases, the number of points at which the approximation is sought likewise steadily increases.

When $q=0$, the rational fraction becomes the familiar interpolation-polynomial of *Lagrange*,

$$\sum_{1}^{i=p+q+1} \frac{f(a_i)}{\phi'(a_i)} \cdot \frac{\phi(x)}{x-a_i},$$

in which

$$\phi(x) = (x-a_1)(x-a_2)\cdots(x-a_{p+q+1}).$$

This has been put into a very interesting form by *Frobenius* [95] which permits, without reconstruction,† of an indefinite increase in the number of its terms. Let us first take $1/(z-x)$ as the particular function of x for which an approximation is sought. From the equations

*The only investigation of this character is found in [76], but on account of the nature of the functions there considered certain variations were made in the construction of the table.

†Cf. also [99, *a*].

$$\frac{1}{z-x}=\frac{1}{(z-a_1)-(x-a_1)}=\frac{1}{z-a_1}+\frac{x-a_1}{z-a_1}\cdot\frac{1}{z-x}$$

$$=\frac{1}{z-a_1}+\frac{x-a_1}{z-a_1}\left(\frac{1}{z-a_2}+\frac{x-a_2}{z-a_2}\cdot\frac{1}{z-x}\right)=\cdots,$$

the series

$$(1)\quad \frac{1}{z-x}=\frac{1}{z-a_1}+\frac{x-a_1}{(z-a_1)(z-a_2)}+\frac{(x-a_1)(x-a_2)}{(z-a_1)(z-a_2)(z-a_3)}+\cdots,$$

is immediately derived, provided that the a_i are so distributed as to fulfill proper conditions for the convergence of the series. If now we take successively 1, 2, 3, $\cdots$ terms of the expansion, we obtain the series of polynomials,

$$N_0(x)=\frac{1}{z-a_1},\quad N_1(x)=\frac{1}{z-a_1}+\frac{x-a_1}{(z-a_1)(z-a_2)},\ \cdots,$$

and it is evident that $N_n(x)$ for the $n+1$ values $x=a_1, a_2, \cdots, a_{n+}$ agrees in value with $1/(z-x)$. By applying to (1) the well-known formula of *Euler* [1, *a*]* for converting any infinite series into a continuous fraction it follows immediately that these polynomials are the successive convergents of the continued fraction

$$\cfrac{\frac{1}{z-a_1}}{1}\ \cfrac{\frac{x-a_1}{z-a_2}}{-1+\frac{x-a_1}{z-a_2}}\ \cfrac{\frac{x-a_2}{z-a_3}}{-1+\frac{x-a_2}{z-a_3}-}\cdots.$$

The generalization of formula (1) can be made at once in the familiar manner by the use of *Cauchy's* integral. We get thus

$$f(x)=\frac{1}{2i\pi}\int\frac{f(z)dz}{z-x}=f(a_1)+\frac{(x-a_1)}{2i\pi}\int\frac{f(z)dz}{(z-a_1)(z-a_2)}+\cdots,$$

which by placing

$$\phi_n(x)=(x-a_1)(x-a_2)\cdots(x-a_n)$$

may be written

* Cf. *Encyklopädie der Math. Wiss.*, I A 3, p. 134, formula (104).

$$(2) \qquad f(x) = f(a_1) + (x - a_1)\left(\frac{f(a_1)}{\phi_2'(a_1)} + \frac{f(a_2)}{\phi_2'(a_2)}\right) + \cdots$$

$$\cdots + (x - a_1)(x - a_2)\cdots(x - a_n)\sum_{i=1}^{n+1}\frac{f(a_i)}{\phi_{n+1}'(a_i)} + \cdots.$$

For most interesting discussions of the convergence and properties of series having the form

$$A_0 + A_1(x - a_1) + A_2(x - a_1)(x - a_2) + \cdots$$

I may refer to memoirs by *Frobenius* [95] and *Bendixson* [99, c]. I shall content myself here with pointing out one simple application which is given implicitly by both writers but has been noted again recently by *Laurent* [103].

Let $f(x)$ be any analytic function the values of which are given at a series of points p_i having a regular point P as their limit. Describe about P as center any circle C within and upon which $f(x)$ is holomorphic, and denote the points p_i which fall within this circle by $a_1, a_2, \cdots$. Then $\lim a_i = P$. If now z describes the perimeter of the circle and x is a fixed interior point, the series (1) will be uniformly convergent and consequently permit of integration term by term. Equation (2) therefore gives an expression for $f(x)$ which is valid in the interior of C. This expression shows at once that an analytic function is determined uniquely when its values are known in a sequence of points having a regular point P as their limit. If, in particular, each $f(a_i) \equiv 0$, $f(x)$ must vanish identically. In other words, the zeros of an analytic function can not be infinitely dense in the vicinity of a non-singular point. Further, *Bendixson* points out that the convergence of the right hand member of (2) is not only the necessary but the sufficient condition that $f(a_1)$, $f(a_2)$, $f(a_3)$, $\cdots$ shall be the values of some analytic function at a set of points a_i having a limit point P.

We turn now to the generalization of the algorithm of the continued fraction. The first investigation on this subject is found in a paper of *Jacobi*,* published posthumously in 1868. The developments of *Jacobi* were, however, of a purely numerical nature. On this side they have been perfected recently by *Fr. Meyer* [83]. The first example of a functional extension was given by *Hermite* in his famous memoir [84] upon the transcendence of e, and the theory has been developed since independently of each other by *Pincherle* and *Padé*.

To explain the nature of the generalization it will be desirable first to refer to the mode in which a continued fraction is commonly generated. Two numbers or functions, f_0 and f_1, are given, from which a sequence of other numbers or functions is obtained by placing

$$
\begin{aligned}
f_2 &= \lambda_1 f_1 - f_0, \\
f_3 &= \lambda_2 f_2 - f_1, \\
f_4 &= \lambda_3 f_3 - f_2, \\
&\cdots\cdots
\end{aligned}
\tag{3}
$$

in which the λ_i are determined in accordance with some stated law. For the quotient f_0/f_1, we obtain successively

$$\frac{f_0}{f_1} = \lambda_1 - \frac{1}{\dfrac{f_1}{f_2}} = \lambda_1 - \frac{1}{\lambda_2 - \dfrac{1}{\dfrac{f_2}{f_3}}} = \cdots,$$

and it therefore gives rise to the continued fraction

$$\lambda_1 - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} - \cdots. \tag{4}$$

By means of the equations (3) each f_{n+1} can be expressed linearly in terms of the initial quantities f_0, f_1. Thus

$$f_{n+1} = A_{1,\,n+1} f_1 + A_{0,\,n+1} f_0, \tag{5}$$

in which $A_{0,\,n+1}$, $A_{1,\,n+1}$ are polynomials in the elements λ_i. It is easy to see that these polynomials both satisfy the same difference

* "Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird." *Journ. für Math.*, vol. 69 (1868), p. 29.

equation as f_i,

$$f_{n+1} = \lambda_n f_n - f_{n-1};$$

and for their initial values we have

$$A_{1,1} = 1, \quad A_{0,1} = 0,$$

$$A_{1,2} = \lambda_1, \quad A_{0,2} = -1.$$

Consequently $A_{1,n}$ and $-A_{0,n}$ are the numerator and denominator of the $(n-1)$th convergent of (4).

When the generating relations have the form

$$f_0 = \lambda_1 f_1 + \mu_2 f_2,$$

$$f_1 = \lambda_2 f_2 + \mu_3 f_3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

the resultant continued fraction is

$$(4') \qquad \lambda_1 + \frac{\mu_2}{\lambda_2 +} \frac{\mu_3}{\lambda_3 +} \cdots.$$

A distinction then appears between the system of functions $(A_{1,n+1}, -A_{0,n+1})$ and the system which consists of the numerator and denominator of the nth convergent. Though the quotient of the two functions of either system is the nth convergent, the former pair of functions satisfy the same relation of recurrence as the f_i, namely,

$$f_n = \lambda_{n+1} f_{n+1} + \mu_{n+2} f_{n+2};$$

while the corresponding relation for the other system is

$$g_n = \lambda_n g_{n-1} + \mu_n g_{n-2}.$$

The latter equation is called by *Pincherle* [77, a] the inverse of the former. In the continued fraction (4) we took $\mu_i = -1$ so that the two relations were coincident.

The immediate generalization of these considerations is obtained by taking $m+1$ initial quantities $f_0, f_1, \cdots, f_m$ in place of two. With a very slight change of notation we may write

$$
(6) \qquad \begin{aligned}
f_0 + \lambda_1 f_1 + \mu_2 f_2 + \cdots + \nu_m f_m &= f_{m+1}, \\
f_1 + \lambda_2 f_2 + \mu_3 f_3 + \cdots + \nu_{m+1} f_{m+1} &= f_{m+2}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
f_{n-m} + \lambda_{n-m+1} f_{n-m+1} + \mu_{n-m+2} f_{n-m+2} + \cdots + \nu_n f_n &= f_{n+1}.
\end{aligned}
$$

Then by expressing f_n in terms of the $m+1$ given quantities we have

$$
(7) \qquad f_n = A_{0,n} f_0 + A_{1,n} f_1 + \cdots + A_{m,n} f_m,
$$

in which $A_{i,n}$ is a polynomial in terms of the $\lambda_i, \mu_{i+1}, \cdots, \nu_{i+m-1}$ $(i = 1, 2, \cdots, n-m)$. These $m+1$ polynomials $A_{i,n}$ satisfy the same difference equation (6) as the f_n, and for their initial values we plainly have

$$
\begin{array}{cccccc}
 & A_{0,n} & A_{1,n} & \cdots & A_{m,n} \\
n = 0 & 1 & 0 & \cdots & 0, \\
n = 1 & 0 & 1 & \cdots & 0, \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
n = m & 0 & 0 & \cdots & 1.
\end{array}
$$

Hence they constitute a complete system of independent integrals of (6). Furthermore, in analogy with the relation between two successive convergents of (4),

$$
\begin{vmatrix} D_n & N_n \\ D_{n-1} & N_{n-1} \end{vmatrix} = 1,
$$

we have [83, a, p. 170]

$$
(8) \qquad \begin{vmatrix}
A_{0,n} & A_{1,n} & \cdots & A_{m,n} \\
A_{0,n+1} & A_{1,n+1} & \cdots & A_{m,n+1} \\
\cdot & \cdot & \cdot & \cdot \\
A_{0,n+m} & A_{1,n+m} & \cdots & A_{m,n+m}
\end{vmatrix} = (-1)^{nm}.
$$

The relation which is the inverse of (6) has the form

$$
(9) \qquad g_{n+m} + \lambda_n g_{n+m-1} + \mu_n g_{n+m-2} + \cdots + \nu_n g_n = g_{n-1}.
$$

To obtain a system of independent integrals of this equation, let

$P_{0,n}$ denote the minor of $A_{0,n}$ in (8), $P_{1,n}$ the minor of $A_{1,n}$ after the first column has been moved over the remaining columns so as to become the last, $P_{2,n}$ the minor of $A_{2,n}$ after the first two columns have been moved over the remaining columns so as to become the last two, and so on. It can be demonstrated easily that the desired system is obtained by placing $g_{i,n+m} \equiv P_{i,n}$ $(i = 0, 1, \cdots, m)$, and these new polynomials rather than the $A_{i,n}$ are the true analogues of the numerator and denominator of an ordinary continued fraction. The connection between the two systems of polynomials is, however, both an intimate and a reciprocal one, for not only is (9) the inverse of (6) but the converse is also true. On this account the two systems can be employed simultaneously with advantage in working with the generalized continued fraction.

For all except the very lowest values of n the new polynomials can be found from the equations*

$$(9') \quad P_{i,n} + \lambda_n P_{i,n-1} + \mu_n P_{i,n-2} + \cdots + \nu_n P_{i,n-m} = P_{i,n-m-1}.$$

In place of these relations it will be often found convenient to employ such a process as is indicated in the following equations for $m = 2$ [83, a, p. 180].†

$$\frac{P_{0,1}}{P_{1,1}} = q_{1,1}, \quad \frac{P_{0,2}}{P_{1,2}} = q_{1,1} + \frac{q_{2,2}}{q_{2,1}}, \quad \frac{P_{0,3}}{P_{1,3}} = q_{1,1} + \frac{q_{2,2} + \dfrac{1}{q_{3,1}}}{q_{2,1} + \dfrac{q_{3,2}}{q_{3,1}}},$$

$$\frac{P_{0,4}}{P_{1,4}} = q_{1,1} + \frac{q_{2,2} + \dfrac{1}{q_{3,1} + \dfrac{q_{4,2}}{q_{4,1}}}}{q_{2,1} + \dfrac{q_{3,2} + \dfrac{1}{q_{4,1}}}{q_{3,1} + \dfrac{q_{4,2}}{q_{4,1}}}}, \cdots, \qquad \begin{matrix}(q_{i,1} = -\lambda_i) \\ (q_{i,2} = -\mu_i)\end{matrix}$$

* Cf. [83, a, p. 174, eq. X].

† Cf. E. Fürstenau, "Ueber Kettenbrüche höherer Ordnung"; *Jahresbericht über das königliche Realgymnasium zu Wiesbaden;* 1873 4. See also Scott's *Determinants*, Chap. 13, § 11–12.

We may therefore very properly call the system of values

$$\begin{bmatrix} \lambda_1 & \mu_1 & \cdots & \nu_1 \\ \lambda_2 & \mu_2 & \cdots & \nu_2 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

the *norm* of a generalized continued fraction, which itself consists of the computation of the $P_{i,n}$ or their ratios.

To apply this generalization to the construction of algebraic continued fractions, it is only necessary to select as the $m+1$ initial functions $f_0, \cdots, f_m$ series in ascending powers or series in descending powers of x. The nature of the ensuing theory will be explained sufficiently by developing here the simplest case, in which three such series are given [77, c.] Take then

$$S_0 = k_0 + \frac{k_1}{x} + \frac{k_2}{x^2} + \cdots \qquad (k_0 \neq 0),$$

$$S_1 = \frac{l_0}{x} + \frac{l_1}{x^2} + \frac{l_2}{x^3} + \cdots \qquad (l_0 \neq 0),$$

$$S_2 = \frac{m_0}{x^2} + \frac{m_1}{x^3} + \frac{m_2}{x^4} + \cdots. \qquad (m_0 \neq 0)$$

If we next place

$$(10) \qquad S_0 + (a_0 x + a_0')S_1 + b_0 S_2 = S_3,$$

the coefficients a_0, a_0', b_0 can be so determined that S_3 shall begin with at least as high a power of $1/x$ as the third. Normally the degree is exactly 3, and similarly for each consecutive value of n we have

$$S_n + (a_n x + a_n')S_{n+1} + b_n S_{n+2} = S_{n+3}$$

in which S_n denotes a series beginning with the nth power of $1/x$. Hence unless certain specified conditions are satisfied, a regular continued fraction will be obtained having the norm:

$$\begin{Bmatrix} 1 & a_0 x + a_0' & b_0 \\ 1 & a_1 x + a_1' & b_1 \\ 1 & a_2 x + a_2' & b_2 \\ \cdot & \cdot & \cdot \end{Bmatrix}.$$

This norm will not be altered in any way by dividing (10) through by S_0. It is therefore determined uniquely by the ratios of S_0, S_1, S_2, and conversely the ratios by the norm.

Without loss of generality we may set $S_0 = 1$. Place also

$$(11) \qquad S_{n+3} = A_{n+3} + B_{n+3}S_1 + C_{n+3}S_2,$$

$$P_n = \begin{vmatrix} B_n & C_n \\ B_{n+1} & C_{n+1} \end{vmatrix}, \quad Q_n = \begin{vmatrix} C_n & A_n \\ C_{n+1} & A_{n+1} \end{vmatrix}, \quad R_n = \begin{vmatrix} A_n & B_n \\ A_{n+1} & B_{n+1} \end{vmatrix}.$$

If then $n+3$ in (11), is replaced successively by n and $n+1$, and the two equations are solved for S_1 and S_2, we obtain

$$S_1 = \frac{Q_n + C_{n+1}S_n - C_n S_{n+1}}{P_n}$$

or

$$(12) \qquad S_1 - \frac{Q_n}{P_n} = \frac{\lambda_n}{P_n} \qquad (\lambda_n = C_{n+1}S_n - C_n S_{n+1}),$$

and

$$(13) \qquad S_2 - \frac{R_n}{P_n} = \frac{\mu_n}{P_n} \qquad (\mu_n = B_n S_{n+1} - B_{n+1}S_n).$$

An examination of P_n, Q_n, R_n, λ_n, μ_n will show that their degrees in x are

$$n-1,\ n-2,\ n-3,\ -r-1,\ -r. \qquad (n = 2r),$$

$$n-1,\ n-2,\ n-3,\ -r-1,\ -r-1 \qquad (n = 2r+1).$$

Hence the expansions of Q_n/P_n and R_n/P_n in descending powers of x, agree with S_1 and S_2 to terms of degree $3r-1$ and $3r-2$ inclusive if $n = 2r$, and of the $3r$th degree if $n = 2r+1$. The generalized continued fraction therefore affords a solution of the problem: to find two rational fractions with a common denominator which shall give as close an approximation to the given functions S_1 and S_2 as is consistent with the degrees prescribed for their numerators and denominators.

When three series in ascending powers of x,

$$\bar{S}_i = k_0^{(i)} + k_1^{(i)}x + k_2^{(i)}x^2 + \cdots \qquad (i = 1, 2, 3),$$

are chosen as the initial functions, a more comprehensive algorithm can be introduced. *Padé* [79, *a*] takes three polynomials $A_p^{(1)}(x)$, $A_{p'}^{(2)}(x)$, $A_{p''}^{(3)}(x)$ with undetermined coefficients, the degrees of which are indicated by their subscripts, and requires that their coefficients shall be so determined that the expansion of

$$A_p^{(1)}\overline{S}_1 + A_{p'}^{(2)}\overline{S}_2 + A_{p''}^{(3)}\overline{S}_3$$

in ascending powers of x shall begin with as high a power as possible. Ordinarily this is the $(p + p' + p'' + 2)$th power. To each set of values of p, p', p'' he shows that there corresponds uniquely a group of three polynomials which he terms the "associated polynomials," and these groups he arranges into a table of triple entry according to the values of p, p', p''. An exactly similar table can not be constructed for three series in descending powers of x, inasmuch as the substitution of $1/x$ for x in $A_p^{(1)}, \cdots, A_{p''}^{(2)}$ gives three rational fractions, with powers of x in the denominators which can not be thrown away unless

$$p = p' = p''.$$

The new table is handled by *Padé* in the same manner as the one previously constructed for a single series. In particular, he examines the relations

$$\alpha A_p^{(i)} + \beta A_q^{(i)} + \gamma A_r^{(i)} = A_s^{(i)} \qquad (i = 1, 2, 3),$$

which exist between four successive groups of associated polynomials, α, β, γ being rational functions of x which are independent of the value of i. When it is possible to so select a sequence $\cdots A_p^{(i)}, A_q^{(i)}, A_r^{(i)}, A_s^{(i)}, A_t^{(i)}, \cdots$ that α, β, γ, are polynomials of invariable degree for any four consecutive terms in the sequence, the sequence or continued fraction is said to be regular. In a normal table there are found to be four distinct types of such continued fractions. It is worth noting, however, that the diagonal type which was the best in an ordinary table, no longer exists since it is found that when the sequence fills a diagonal file of the table, α, β, and γ are no longer polynomials but rational fractions having a common denominator.

In one important respect *Padé's* investigation has a narrower reach than *Pincherle's* and needs completion. The existence of a second group of associated polynomials — the P_n, Q_n, R_n of *Pincherle* — is not brought to light. As has been already pointed out, it is this second group of polynomials which is the true analogue of the convergent of an ordinary continued fraction and which must take precedence in considering the convergence of the algorithm or the closeness of the approximation afforded to the initial functions. *Pincherle's* definition of convergence [82] is not, however, so framed as to require explicitly the introduction of these polynomials. If the given system of difference equations is

$$f_{n+3} = c_n f_{n+2} + d_n f_{n+1} + f_n \qquad (n = 0, 1, 2, \cdots), \tag{14}$$

the continued fraction is said by him to be convergent when the two following conditions are fulfilled :

(1) There is a system of integrals F_n, F'_n, F''_n of (14) such that F'_n/F_n, F''_n/F_n have limits for $n = \infty$, and these limits are different from 0.

(2) There is also one particular integral — called by *Pincherle* the *integrale distinto* — the ratio of which to every other integral of (14) has the limit zero.

Pincherle's interest is evidently concentrated upon this *principal* integral. It seems to me, however, more natural to call the algorithm convergent when the ratios Q_n/P_n and R_n/P_n (cf. Equations 12 and 13) converge to finite limits for $n = \infty$. Under ordinary circumstances these limits will doubtless coincide with the ratios of the generating functions, f_1/f_0 and f_2/f_0.

In the case of an ordinary continued fraction the two definitions coalesce. For suppose that the nth convergent N_n/D_n of (4′) has the limit L. Then $N_n - LD_n$ is such an integral of the difference equation,

$$f_n = \lambda_n f_{n-1} + \mu_n f_{n-2},$$

that its ratio to any other integral, $k_1 N_n + k_2 D_n$, has the limit 0. Conversely, if the principal integral $N_n - LD_n$ exists, there must be a limit L for the continued fraction. Possibly the case in which the principal integral is D_n might be called an exception, since the continued fraction is then convergent by *Pincherle's* definition, but $\lim N_n/D_n = \infty$.

A study of the conditions of convergence, so far as I am aware, has at present been made in only two special cases. *Fr. Meyer* [83, *a*, § 7] has made a *partial* investigation when the coefficients $\lambda_n, \cdots, \nu_n$ in equations (6) are negative constants. *Pincherle* [82] has examined the case in which the coefficients of the recurrent relation

$$f_n + (a_n x + a'_n) f_{n+1} + b_n f_{n+2} = f_{n+3}$$

have limiting values and finds that the generalized continued fraction is convergent for sufficiently large values of x. Let the limits of the coefficients be denoted by a, a', and b respectively. To demonstrate the convergence he avails himself of the notable theorem of *Poincaré*, already cited in Lecture 4. If, namely, no two roots of the equation

$$z^3 - bz^2 - (ax + a')f - 1 = 0 \tag{15}$$

are of equal modulus, f_n/f_{n-1} will have a limit for $n = \infty$, and this limit will be one of the roots of the auxiliary equation (15), usually the root of greatest modulus. From this it follows directly that A_n/A_{n-1}, B_n/B_{n-1}, C_n/C_{n-1} as quotients of integrals of the difference equation last given, also P_n/P_{n-1}, Q_n/Q_{n-1}, R_n/R_{n-1} as integrals of the inverse equation, have each a definite limit. The existence of limits for Q_n/P_n and of R_n/P_n is then established for sufficiently great values of x, and the analytic character of these limits is finally argued. Let them be denoted by $U(x)$ and $V(x)$. Then $X_n = A_n + B_n U(x) + C_n V(x)$ is the principal integral of the difference equation, and has the following distinctive property: Its expansion in powers of $1/x$ begins with the highest possible power consistent with the degrees of A_n, B_n, C_n, and coincides with f_n for each successive value of n.

Bibliography of Memoirs relating to Algebraic Continued Fractions.

In the following bibliography only works in Latin, Italian, French, German, and English are included. In *Wölffing's Mathematischer Bücherschatz* (heading *Kettenbrüche*) several dissertations, etc., are mentioned which may possibly relate to algebraic continued fractions but which are not accessible to the writer. They are therefore not included here. The writer would be glad to have his attention called to any noteworthy omissions in the bibliography.

In many cases it has been extremely difficult to draw the line between inclusion and exclusion, especially under divisions VI–IX.

Any classification of the material which may be adopted will be open to objections, but even an imperfect classification will probably add greatly to the usefulness of the bibliography. Since much of the work relating to algebraic continued fractions appears elsewhere under other headings, it is believed that such a bibliography as is here given may be of service.

For a brief resumé of the theory of algebraic continued fractions the reader is referred to *Osgood's* section of the *Encyklopädie der Math. Wissenschaft,* II B 1, §§ 38–39.

I. On the Derivation of Continued Fractions from Power Series. General Theory.

A. Early Works.

1. **Euler.** (*a*) Introductio in analysin infinitorum, vol. 1, chap. 18, 1748.

(*b*) De transformatione serierum in fractiones continuas. Opuscula analytica, vol. 2, pp. 138–177, 1785.

2. **Lambert.** (*a*) Verwandlung der Brüche. Beyträge zum Gebrauche der Mathematik und deren Anwendung, vol. 2_1, p. 54 ff., p. 161, 1770.

(*b*) Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques. Histoire de l'Acad. roy. des sciences et belles-lettres à Berlin, 1768.

3. **Trembley.** Recherches sur les fractions continues. Mém. de l'Acad. roy. de Berlin, 1794, pp. 109–142.

4. **Kausler.** (*a*) Expositio methodi series quascunque datas in fractiones continuas convertendi. Mém. de l'Acad. imp. des sciences de St. Pétersbourg, vol. 1, pp. 156–174, 1802.

(*b*) De insigni usu fractionum continuarum in calculo integrale. Ibid., vol. 1, pp. 181–194, 1803.

5. **Viscovatov.** (*a*) De la méthode générale pour reduire toutes sortes des quantités en fractions continues. Ibid., vol. 1, pp. 226–247, 1805.

(*b*) Essai d'une méthode générale pour réduire toutes sortes de séries en fractions continues. Nova Acta Acad. Scient. imp. Petropolitanæ, vol. 15, pp. 181–191, 1802.

6. **Bret.** Théorie générale des fractions continues. Gergonne's Annales de Math., vol. 9, pp. 45–49, 1818. Unimportant.

7. **Scubert.** De transformatione seriei in fractionem continuam. Mém. de l'Acad. imp. des sciences de St. Pétersbourg, vol. 7, pp. 139–158, 1820.

8. **Stern.** (*a*) Zur Theorie der Kettenbrüche und ihre Anwendung. Jour. für Math., vol. 10, pp. 241–265, 1833.

(*b*) Zur Theorie der Kettenbrüche. Jour. für Math., vol. 18, pp. 69–74, 1838.

9. **Heilermann.** (*a*) Ueber die Verwandlung der Reihen in Kettenbrüche. Jour. für Math., vol. 33, pp. 174–188, 1846; also vol. 46, pp. 88–95, 1853.

(*b*) Zusammenhang unter den Coefficienten zweier gleichen Kettenbrüche von verschiedener Form. Zeitschrift für Math. und Phys., vol. 5, pp. 362–363, 1860. Unimportant.

10. **Hankel.** Ueber die Transformation von Reihen in Kettenbrüche. Berichte der Sächischen Gesellschaft der Wissenschaft zu Leipzig, vol. 14, pp. 17–22, 1862.

11. **Muir.** (*a*) On the transformation of *Gauss*' hypergeometric series into a continued fraction. Proc. of the London Math. Soc., vol. 7, pp. 112–118, 1876.

(*b*) New general formulæ for the transformation of infinite series into continued fractions. Trans. of the R. Soc. of Edinburgh, vol. 27, pp. 467–471, 1876.

The general formulæ in these memoirs, which Muir supposed to be new, had been previously given by Heilermann in 9(*a*).

12. **Heine.** Handbuch der Kugelfunction, 2[te] Auflage, 1878; chap. 5, Die Kettenbrüche, pp. 260–297.

This gives a good idea of the state of the theory up to 1878.

B. The Modern Theory.

The beginnings of this theory are to be found in Nos. 110 and 111.

13. **Frobenius.** Ueber Relationen zwischen den Näherungsbrüchen von Potenzreihen. Jour. für Math., vol. 90, pp. 1–17, 1881.

This fundamental memoir marks an important advance. See 16(*a*).

14. **Stieltjes.** Sur la réduction en fraction continue d'une série procédant suivant les puissances descendantes d'une variable. Ann. de Toulouse, vol 3, H, pp. 1–17, 1889.

15. **Pincherle.** Sur une application de la théorie des fractions continues algébriques. Comp. Rend., vol. 108, p. 888, 1889.

16. **Padé.** (*a*) Sur la représentation approchée d'une fonction par des fractions rationnelles. Thesis, published in the Ann. de l'Ec. Nor., ser. 3, vol. 9, supplement, pp. 1–93, 1892.

This very fundamental memoir is the best one to read for the purpose of learning the elements of the theory of algebraic continued fractions. The same point of view is taken as by Frobenius in (13) and is more completely developed. The thesis was preceded by the two following preliminary notes :

(a') Sur la représentation approchée d'une fonction par des fractions rationnelles. Comp. Rend, vol. 111, p. 674, 1890.

(a'') Sur les fractions continues régulières relatives a e^x. Comp. Rend, vol. 112, p. 712, 1891.

(*b*) Recherches nouvelles sur la distribution des fractions rationnelles approchées d'une fonction. Ann. de l'Ec. Nor., ser. 3, vol. 19, pp. 153–189, 1902.

(*c*) Aperçu sur les développements récents de la théorie des fractions continues. Compte rendu du deuxième Congrès international des mathématiciens tenu a Paris, pp. 257–264, 1900.

Only a restricted portion of the field is here reviewed, and in this portion the important work of Pincherle is overlooked.

17. **Padé.** (*a*) Sur les séries entières convergentes ou divergentes et les fractions continues rationelles. Acta Math., vol. 18, pp. 97–111, 1894.

(a') Sur la possibilité de définir une fonction par une série entière divergente. Comp. Rend., vol. 116, p. 686, 1893.

See also No. 26*a*, 76.

II. On Convergence.

(For a résumé of the criteria for the convergence of continued fractions with real elements see Pringsheim's report in the *Encyklopädie der mathematischen Wissenschaften*, I A 3, p. 126, ff.)

18. **Riemann.** Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita, 1863. Gesammelte mathematische Werke, pp. 400–406.

18, *bis.* **Worpitzky.** Untersuchung über die Entwickelung der monodromen und monogenen Functionen durch Kettenbrüche. Programm, Friedrichs Gymnasium und Realschule, Berlin, 1865.

This program and the two following memoirs of Thomé were published before Riemann's posthumous fragment.

19. **Thomé.** (*a*) Ueber die Kettenbruchentwickelung der Gauss'schen Function $F(\alpha, 1, \gamma, x)$. Jour. für Math., vol. 66, pp. 322–336, 1866.

(*b*) Ueber die Kettenbruchentwickelung des Gauss'schen Quotienten

$$\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}.$$

Ibid., vol. 67, pp. 299–309, 1867.

20. **Laguerre.** Sur l'integrale

$$\int_x^\infty \frac{e^{-x}}{x}\,dx.$$

Bull. de la Soc. Math. de France, vol. 7, pp. 72–81, 1879, or Oeuvres, vol. 1, p. 428.

Historically an important memoir because of its development of the connection between a divergent power series and convergent continued fraction. See the first footnote in lecture 4; also No. 102, p. 30.

21. **Halphen.** (*a*) Sur la convergence d'une fraction continue algébrique. Comp. Rend., vol. 100 (1885), pp. 1451–1454.

(*b*) Same subject. Ibid., vol. 106 (1888), pp. 1326–1329.

(*c*) Traité des fonctions elliptiques. Chap. 14. Fractions continues et intégrales pseudo-elliptiques.

22. **Pincherle.** Alcuni teoremi sulle frazioni continue. Atti delle R. Accad. dei Lincei, ser. 4, vol. 5_1, pp. 640–643, 1889.

The test for convergence given here is included in a more general criterion given later by Pringsheim, No. 29.

23. **Pincherle.** Sur les fractions continues algébriques. Ann. de l'Ec. Nor., ser. 3, vol. 6, pp. 145–152, 1889.

An incomplete result is here obtained. See No. 32*c* for the complete theorem.

24. **Padé.** Sur la convergence des fractions continues simples. Comp. Rend., vol. 112, p. 988, 1891. Also found in §§ 45–47 of No. 16*a*.

25. **Banning.** Ueber Kugel- und Cylinderfunktionen und deren Kettenbruchentwickelung. Dissertation, Bonn, 1894, pp. 1–33.

26. **Stieltjes.** (a) Recherches sur les fractions continues. Annales de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47. 1894–95. Published also in vol. 32 of the Mémoires présentés à l'Acad. des sciences de l'Institut National de France.

A rich memoir, developing particularly the connection between an important class of continued fractions and the corresponding integrals.

(a') Sur un développement en fraction continue. Comp. Rend., vol. 99, p. 508, 1884.

(a'') Same subject. Ibid., vol. 108 (1889), p. 1297.

(a''') Sur une application des fractions continues. Ibid., vol. 118 (1894), p. 1315.

(a^{iv}) Recherches sur les fractions continues. Ibid., vol. 118 (1894), p. 1401.

Markoff. (b) Note sur les fractions continues. Bull. de l'Acad. imp. des sciences de St. Petersbourg, ser. 5, vol. 2, pp. 9–13, 1895.

This gives a discussion of the relation of his work to that of Stieltjes.

27. **H. von Koch.** (a) Sur un théoréme de *Stieltjes* et sur les fonctions définies par des fractions continues. Bull. de la Soc. Math. de France, vol. 23, pp. 33–40, 1895.

(a') Sur la convergence des determinants d'ordre infini et des fractions continues. Comp. Rend., vol. 120, p. 144, 1895.

28. **Markoff.** Deux démonstrations de la convergence de certaines fractions continues. Acta Math., vol. 19, pp. 93–104, 1895.

Contained also in his Differenzenrechnung (deutsche Uebersetzung), chap. 7, § 21–22.

This discusses the convergence of the usual continued fraction for

$$\int_a^b \frac{f(y)dy}{z-y}$$

when $f(y) > 0$ between the limits of integration.

29. **Pringsheim.** Ueber die Convergenz unendlicher Kettenbrüche. Sitzungsberichte der math.-phys. Classe der k. bayer'schen Akad. der Wissenschaften, vol. 28, pp. 295–324, 1898.

The most comprehensive criteria for convergence yet obtained are found in 29, 31, and 32b.

30. **Bortolotti.** Sulla convergenza delle frazioni continue algebriche. Atti della R. Accad. dei Lincei, ser. 5, vol. 8_1, pp. 28–33, 1899.

31. **Van Vleck.** On the convergence of continued fractions with complex elements. Trans. Amer. Math. Soc., vol. 2, pp. 215–233, 1901.

32. **Van Vleck.** (*a*) On the convergence of the continued fraction of Gauss and other continued fractions. Annals of Math., ser. 2, vol. 3, pp. 1–18, 1901.

(*b*) On the convergence and character of the continued fraction

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots$$

Trans. Amer. Math. Soc., vol. 2, pp. 476–483, 1901.

(*c*) On the convergence of algebraic continued fractions whose coefficients have limiting values. Ibid., vol. 5, pp. 253–262, 1904.

33. **Montessus.** (*a*) Sur les fractions continues algébriques. Bull. de la Soc. Math. de France, vol. 30, pp. 28–36, 1902.

The content of this memoir was discussed in lecture 5.

(*b*) Same title. Comp. Rend., vol. 134 (1902), p. 1489.

See also 37*a*′, 41.

III. On Various Continued Fractions of Special Form.

A. The Continued Fraction of Gauss.

34. **Gauss.** Disquisitiones generales circa seriem infinitam

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots$$

Deutsche Uebersetzung von Simon, or Werke, vol. 3, pp. 134–138, 1812.

34, *bis.* **Vorsselman de Herr.** Specimen inaugurale de fractionibus continuis. Dissertation, Utrecht, 1833.

Numerous references are given here to the early literature upon continued fractions.

34, *ter.* **Heine.** Auszug eines Schreibens über Kettenbrüche von Herrn E. Heine an den Herausgeber. Jour. für Math., vol. 53, pp. 284–285, 1857.

See also 40*c*, p. 231.

35. **Euler.** (*a*) Commentatio in fractionem continuam in qua illustris *Lagrange* potestates binomiales expressit. Mémoires de l'Acad. imp. des sciences de St. Pétersbourg, vol. 6, pp. 3–11, 1818.

Padé. (*b*) Sur la généralisation des développements en fractions continues, donnés par *Gauss* et par *Euler*, de la fonction $(1+x)^m$. Comp. Rend., vol. 129, p. 753, 1899.

(*c*) Sur la généralisation des développements en fractions continues, donnés par *Lagrange* de la fonction $(1+x)^m$. Ibid., vol. 129, p. 875, 1899.

(*d*) Sur l'expression générale de la fraction rationnelle approchée de $(1+x)^m$. Ibid., vol. 132, p. 754, 1901.

See also Nos. 11, 32*a*, 65.

B. The Continued Fractions for e^x.

36. **Winckler.** Ueber angenäherte Bestimmungen. Wiener Berichte, Math.-naturw, Classe, vol. 72, pp. 646–652, 1875.

37. **Padé.** (*a*) Mémoire sur les développements en fractions continues de la fonction exponentielle, pouvant servir d'introduction à la théorie des fractions continues algébriques. Ann. de l'Ec. Nor., Ser. 3, vol. 16, pp. 395–426, 1899.

(*a′*) Sur la convergence des réduites de la fonction exponentielle. Comp. Rend., vol. 127, p. 444, 1898.

See also Nos. 16*a″*, 106, and pages 243–5 of 40*c*.

C. The Continued Fraction of Bessel.

38. **Günther.** Bemerkungen über Cylinder-Functionen. Archiv der Math. und Phys., vol. 56, pp. 292–297, 1874.

39. **Graf.** (*a*) Relations entre la fonction Bessélienne de 1^re espèce et une fraction continue. Annali di Mat., ser. 2, vol. 23, pp. 45–65, 1895.

Giving references to earlier works where the continued fraction of Bessel is found.

Crelier. (*b*) Sur quelques propriétés des fonctions Besséliennes, tirées de la théorie des fractions continues. Annali di Mat., vol. 24, pp. 131–163, 1896.

See also Nos. 25, 32*a*.

D. The Continued Fraction of Heine.

40. **Heine.** (*a*) Ueber die Reihe

$$1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}x^2+\cdots.$$

Jour. für Math., vol. 32, pp. 210–212, 1846.

(*b*) Untersuchung über die (selbe) Reihe. Ibid., vol. 34, pp. 285–328, 1847.

(*c*) Ueber die Zähler und Nenner der Näherungswerthe von Kettenbrüche. Ibid., vol. 57, pp. 231–247, 1860.

Christoffel (*d*) Zur Abhandlung "Ueber Zähler und Nenner" (u. s. w.) des vorigen Bandes. Ibid., vol. 58, pp. 90–91, 1861.

41. **Thomae.** Beiträge zur Theorie der durch die Heine'sche Reihe darstellbaren Funktionen. Jour. für Math., vol. 70, 1869. See pp. 278–281 where the convergence of Heine's continued fraction is proved.

See also 32*a*.

42. (On *Eisenstein's* continued fractions).

Heine. (*a*) Verwandlung von Reihen in Kettenbrüche. Jour. für Math., vol. 32, pp. 205–209, 1846.

See also vol. 34, p. 296.

Muir. (*b*) On Eisenstein's continued fractions. Trans. Roy. Soc. of Edinburgh, vol. 28, part 1, pp. 135–143, 1877.

Muir plainly was not aware of the preceding memoir by Heine.

E. The Continued Fraction of Stieltjes. (*See No. 26.*)

43. **Borel.** Les séries de *Stieltjes*, Chap. 5 of his Mémoire sur les séries divergentes. Ann. de l'Ec. Nor., ser. 3, vol. 16, pp. 107–128; and also chap. 2 of his treatise, Les Séries divergentes, pp. 55–86, 1901.

44. **Padé.** Sur la fraction continue de *Stieltjes*. Comp. Rend., vol. 132, p. 911, 1901.

45. **Van Vleck.** On an extension of the 1894 memoir of *Stieltjes*. Trans. Amer. Math. Soc., vol. 4, pp. 297–332, 1903.

See also Nos. 27, 102.

F. The Continued Fraction for

$$1 + mx + m(m+n)x^2 + m(m+n)(m+2n)x^3 + \cdots$$

and its special cases.

46. **Euler.** (*a*) De seriebus divergentibus. Novi commentarii Acad. scientiarum imperialis Petropolitanæ, vol. 5, pp. 205–237, 1754–5; in particular pp. 225 and 232–237.

(*b*) De transformatione seriei divergentis

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + \cdots$$

in fractionem continuam. Nova acta Acad. scientiarum imperialis Petropolitanæ, vol. 2, pp. 36–45, 1784.

Gergonne. (*c*) Recherches sur les fractions continues. Gergonne's Annales de Math., vol. 9, pp. 261–270, 1818.

47. **Laplace.** (*a*) Traité de mecanique celeste. Oeuvres, vol. 4, pp. 254–257, 1805.

Jacobi. (*b*) De fractione continua in quam integrale $\int_x^\infty e^{-x^2} dx$ evolvere licet. Jour. für Math., vol. 12, pp. 346–347, 1834, or Werke, vol. 6, p. 76.

See also p. 79 of No. 20, and the first note under lecture 2.

G. Periodic Continued Fractions, and Continued Fractions Connected with the Theory of Elliptic Functions.

48. **Abel.** (*a*) Sur l'intégration de la formule différentielle $\rho dx/\sqrt{R}$, R et ρ étant des fonctions entières. Jour. für Math., vol. 1, pp. 185–221, 1826, or Oeuvres, vol 1, p. 104 ff.

Dobnia. (*b*) Sur le développement de $\sqrt{R}$ en fraction continue. Nouvelles Ann. de Math., ser. 3, vol. 10, pp. 134–140, 1891.

49. **Jacobi.** (*a*) Note sur une nouvelle application de l'analyse des fonctions elliptiques à l'algébre. Jour. für Math., vol. 7, pp. 41–43, 1831, or Werke, vol. 1, p. 327.

Borchardt. (*b*) Application des transcendantes abéliennes à la théorie des fractions continues. Ibid., vol. 48, pp. 69–104, 1854.

50. **Tchebychef.** Sur l'intégration des différentielles qui contiennent une racine carrée d'un polynôme du troisième ou du quatrième degré. Mémoires de l'Acad. imp. des sciences de St. Pétersbourg, ser. 6, vol. 8, pp. 203–232, 1857.

51. **Frobenius und Stickelberger.** Ueber die Addition und Multiplication der elliptischen Functionen. Jour. für Math., vol. 88, pp. 146–184, 1880.

52. **Halphen.** Sur les intégrales pseudo-elliptiques. Comp. Rend., vol. 106 (1888), pp. 1263–1270.

53. **Bortolotti.** Sulle frazioni continue algebriche periodiche. Rendiconti del Circolo Mat. di Palermo, vol. 9, pp. 136–149, 1895.

See also Nos. 21, 26(*a*), 40.

H. Miscellaneous.

54. **Euler.** (*a*) Speculationes super formula integrali

$$\int \frac{x^n dx}{\sqrt{a^2 - 2bx + cx^2}}$$

ubi simul egregiæ observationes circa fractiones continuas occurrent. Acta Acad. scientiarum imperialis Petropolitanae, 1784, pars posterior, pp. 62–84, 1782.

(*b*) Summatio fractionis continuæ cujus indices progressionem arithmeticam constituunt. Opuscula Analytica, vol. 2, pp. 217–239, 1785.

55. **Spitzer.** (*a*) Darstellung des unendlichen Kettenbruchs

$$x + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \cdots$$

in geschlossener Form, nebst anderen Bemerkungen. Archiv der Math. und Phys., vol. 30, pp. 81–82, 1858.

(*b*) Darstellung des unendlichen Kettenbruchs

$$2x + 1 + \frac{1}{2x+3} + \frac{1}{2x+5} + \frac{1}{2x+7} + \cdots$$

in geschlossener Form. Ibid., vol. 30, pp. 331–334, 1858.

(*c*) Note über eine Kettenbrüche. Ibid., vol. 33, pp. 418–420, 1859.

(*d*) Darstellung des unendlichen Kettenbruches

$$\Psi(x) = n(2x+1) + \frac{m}{n(2x+3)} + \frac{m}{n(2x+5)} + \cdots$$

in geschlossener Form. Ibid., vol. 33, pp. 474–475, 1859.

56. **Laurent.** (*a*) Note sur les fractions continues. Nouvelles Ann. de Math., ser. 2, vol. 5, pp. 540–552, 1866.

This treats the continued fraction

$$\frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \cdots.$$

E. Meyer. (*b*) Ueber eine Eigenschaft des Kettenbruches $x - \frac{1}{x-} \frac{1}{x-} \cdots$. Archiv der Math. und Phys., ser. 3, vol. 5, p. 287, 1903.

Meyer's results will be found on p. 548 of Laurent's memoir and differs only in that x has been replaced by $-1/x^2$.

57. **Schlömilch.** (*a*) Ueber den Kettenbruch für tan z. Zeitschrift für Math. und Phys., vol. 16, pp. 259–260, 1871.

Glaisher. (*b*) A continued fraction for tan nx. Messenger of Math., ser. 2, vol. 3, p. 137, 1874.

(*c*) Note on continued fractions for tan nx. Ibid., ser. 2, vol. 4, pp. 65–58, 1875.

58. **Schlömilch.** Ueber die Kettenbruchentwickelung für unvollständige Gamma-function. Zeitschrift für Math. und Phys., vol. 16, pp. 261–262, 1871.

This gives the development of $\int_0^x t^{\mu-1} e^{-t} dt$.

59. **Schendel.** Ueber eine Kettenbruchentwickelung. Jour. für Math., vol. 80, pp. 95–96, 1875.

60. **Lerch.** Note sur les expressions qui, dans diverses parties du plan, représentent des fonctions distinctes. Bull. des sciences Math. ser. 2, vol. 10, pp. 45–49, 1886.

61. **Stieltjes.** (*a*) Sur quelques intégrales definies et leur développement en fractions continues. Quar. Jour. of pure and applied Math., vol. 24, pp. 370–382, 1890.

(*b*) Note sur quelques fractions continues. Ibid., vol. 25, pp. 198–200, 1891.

62. **Hermite.** Sur les polynomes de Legendre. Jour. für Math., vol. 107, pp. 80–83, 1891.

This connects $D_x^{(\gamma)} P^{(n)}(x)$ with a continued fraction.

IV. On the Connection of Continued Fractions with Differential Equations and Integrals.

A. Riccati's Differential Equation.

63. **Euler.** (*a*) De fractionibus continuis observationes. Commentarii academiæ scientiarum imperialis Petropolitanæ, vol. 11, see pp. 79–81, 1739.

(*b*) Analysis facilis æquationem Riccatianam per fractionem continuam resolvendi. Mémoires de l' Acad. imperiale des sciences de St. Pétersbourg, vol. 6, pp. 12–29, 1813.

64. **Lagrange.** Sur l'usage des fractions continues dans le calcul intégral. Nouveaux Mém. de l'Acad. roy. des sciences et belles-lettres de Berlin, 1776, pp. 236–264, or Oeuvres, vol. 4, p. 301 ff.

One of the few important early works.

See 54*b*; also No. 66*a* for work on differential equations of the 1st order.

B. Miscellaneous Differential Equations of the Second Order.

In a numerous class of continued fractions the denominators of the convergents satisfy allied (**Heun**, "*gleichgruppige*") differential equations of the second order. Early instances are found in works of *Gauss* (No. 114), *Jacobi* (No. 65) and *Heine* (No. 72). The theory, from two different aspects, is furthest developed in 66*a* and 76.

65. **Jacobi.** Untersuchung über die Differentialgleichung der hypergeometrischen Reihe. Nachlass. Jour. für Math., vol. 56, 1859; see in particular § 8, pp. 160–161, or Werke, vol. 6, p. 184.

66. **Laguerre.** (*a*) Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels. Jour. de Math., ser. 4, vol. 1, pp. 135–165, 1885. Oeuvres vol 2. p 285

This is a comprehensive memoir which incorporates substantially all the following memoirs:

(*b*) Sur la réduction en fractions continues d'une classe assez étendue de fonctions. Comp. Rend., vol. 87 (1878), p. 923, or Oeuvres, vol. 1, p. 322.

(*c*) Same title as (*a*). Bull. de la Soc. Math. de France, vol. 8 (1880), pp. 21–27, or Oeuvres, vol. 1, p. 438.

(*d*) Sur la réduction en fraction continue d'une fraction qui satisfait à une équation linéaire du premier ordre à coefficients rationnels. Comp. Rend., vol. 98 (1884), pp. 209–212 or Oeuvres, vol. 1, p. 445.

67. **Laguerre.** (*a*) Sur l'approximation des fonctions d'une variable au moyen de fractions rationnelles. Bull. de la Soc. Math. de France, vol. 5 (1877), pp. 78–92 or Oeuvres, vol. 1, p. 277.

(*b*) Sur le développement en fraction continue de

$$e^{\operatorname{arc\,tan}\left(\frac{1}{x}\right)} = \int \frac{dx}{1+x^2}.$$

Ibid., vol. 5 (1877), pp. 95–99 or Oeuvres, vol. 1, p. 291.

(*c*) Sur la fonction $\left(\dfrac{x+1}{x-1}\right)^{\omega}$.

Ibid., vol. 8 (1879), pp. 36–52, or Oeuvres, vol. 1, p. 345.

(*d*) Sur la réduction en fractions continues de $e^{F(x)}$, $F(x)$ désignant un polynôme entier. Jour. de Math., ser. 3, vol. 6 (1880), pp. 99–110, or Oeuvres, vol. 1, p. 325.

(*d'*) Same subject. Comp. Rend., vol. 87 (1878), p. 820, or Oeuvres, vol. 1, p. 318.

68. **Humbert.** Sur la réduction en fractions continues d'une classe de fonctions. Bull. de la Soc. Math. de France, vol. 8, pp. 182–187, 1879–1880.

69. **Hermite et Fuchs.** Sur un développement en fraction continue. Acta Math., vol. 4, pp. 89–92, 1884.

See also No. 20, 34 *ter*, 71–76.

C. Differential Equations of Order Higher than the Second.

70. **Pincherle.** Sur la génération de systèmes recurrents au moyen d'une équation linéaire differentielle. Acta Math., vol. 16, pp. 341–363, 1892–3.

See also No. 15, 86, 87, 124*b*.

D. The integral $\int_a^b \frac{f(x)dx}{x-z}$.

71. **Heine.** (*a*) Ueber Kettenbrüche. Monatsberichte der k. preussischen Akad. der Wissenschaften zu Berlin, 1866, pp. 436–451.
(*a′*) Mittheilung über Kettenbrüche. Auszug aus dem Monatsberichte, u. s. w. Jour. für Math., vol. 67, pp. 315-326, 1867.

See also Nos. 12, 26*a*, 28, 45, 102, 113, 118*a*.

E. Hyperelliptic and Similar Abelian Integrals.

72. **Heine.** Die Lamé'schen Functionen verschiedener Ordnungen. Jour. für Math., vol. 60, 1862, pp. 252–303; in particular pp. 256, 275, 294–297. Or see his Handbuch, vol. 1 (2te Auf.), pp. 388–396 and 468.

73. **Laguerre.** Sur l'approximation d'une classe de transcendantes qui comprennent comme cas particulier les intégrales hyperelliptiques. Comp. Rend., vol. 84, pp. 643–645, 1877.
(Not found in vol. 1. of his Oeuvres.)

74. **Humbert.** Sur l'équation différentielle linéaire du second ordre. Jour. de l'Ec. Polytech., vol. 29, cahier 48, pp. 207–220, 1880.

75. **Heun.** (*a*) Die Kugelfunctionen und Lamé'schen Functionen als Determinanten. Dissertation, pp. 1–32, Göttingen, 1881.
(*b*) Ueber lineäre Differentialgleichungen zweiter Ordnung deren Lösungen durch den Kettenbruchalgorithmus verknüpft sind. Habilitationsschrift. 1881.
(*c*) Integration regulärer lineärer Differentialgleichungen zweiter Ordnung durch die Kettenbruchentwickelung von ganzen Abel'schen Integralen dritter Gattung. Math. Ann., vol. 30, pp. 553–560, 1887.
(*d*) Beiträge zur Theorie der Lamé'schen Functionen. Math. Ann., vol. 33, pp. 180–196, 1889.
The important group-properties of the continued fraction are here brought out and are further developed in No. 76.

76. **Van Vleck.** Zur Kettenbruchentwickelung hyperelliptischer und ähnlicher Integrale. Dissertation, Göttingen; published in the Amer. Jour. of Math., vol. 16 (1894), pp. 1–91.

After development first from an algebraic standpoint the subject is carried further by the method of conformal representation. The suggestion of this treatment is given in *Klein's* Differentialgleichungen, 1890–91, vol. 1, pp. 180–186.

V. Generalization of the Algebraic Continued Fraction.

A. General Theory.

So far as I have been able to ascertain, the first instance of the generalization is contained in *Hermite's* memoir, No. 84. The development of a general theory is due to *Pade* and *Pincherle.* Nos. 77*a*, 77*b*, and 79*a* are especially recommended.

77. **Pincherle.** (*a*) Saggio di una generallizzazione delle frazioni continue algebriche. Memoirie della R. Accad. delle Scienze dell' Istituto di Bologna, ser. 4, vol. 10, p. 513–538, 1890.

(*a′*) Di un'estensione dell' algorithmo delle frazioni continue. Rendiconti, R. Istituto Lombardo di Scienze e Lettere, ser. 2, vol. 22, pp. 555–558, 1889.

(*b*) Sulla generalizzazione delle frazioni continue algebrique. Annali di Mat., ser. 2, vol. 19, pp. 75–95, 1891.

78. **Hermite.** Sur la généralisation des fractions continues algébriques. Annali di Mat., ser. 2, vol. 21, pp. 289–308, 1893.

79. **Padé.** (*a*) Sur la généralisation des fractions continues algébriques. Jour. de Math., ser. 4, vol. 10, pp. 291–329, 1894.

(*a′*) Same subject. Comp. Rend., vol. 118, p. 848, 1894.

80. **Bortolotti.** Un contributo alla teoria delle forme lineari alle differenze. Annali di Mat., ser. 2, vol. 23, pp. 309–344, 1895.

81. **Cordone.** Sopra un problema fundamentale delle teoria delle frazioni continue algebriche generalizzate. Rendiconti del Circolo di Palermo, vol. 12, pp. 240–257, 1898.

Cordone seeks the regular algorithms which are similar to those of Padé but occur in connection with n series in descending powers of x.

B. Convergence of the Generalized Algorithm.

82. **Pincherle.** Contributo alla generalizzazione delle frazioni continue. Memoirie della R. Accad. delle Scienze dell' Istituto di Bologna, ser. 5, vol. 4, pp. 297–320, 1894.

83. **W. Franz Meyer.** (*a*) Ueber kettenbruchähnlichen Algorithmen. Verhand. des ersten internationalen Mathematiker-Kongresses in Zürich, pp. 168–181, 1898 ; see in particular § 7.

(a') Zur Theorie der kettenbruchähnlichen Algorithmen. Schriften der phys-ökonomischen Gesellschaft zu Königsberg, vol. 38, pp. 57–66, 1897.

C. Special Cases of the Algorithm.

84. **Hermite.** Sur la fonction exponentielle. Comp. Rend., vol. 77, pp. 18–24, 74–79, 226–233, 285–293, 1873.

This is the famous work proving the transcendence of e.

85. **Hermite.** (a) Sur l'expression $U \sin x + V \cos x + W$. Extrait d'une lettre à Monsieur Paul Gordan. Jour. für Math., vol. 76, pp. 303–311, 1873.

(b) Sur quelques approximations algébriques. Ibid., vol. 76, pp. 342–344, 1873.

(c) Sur quelques équations différentielles linéaires. Extrait d'une lettre à M. L. Fuchs de Gottingue. Ibid., vol. 79, pp. 324–338, 1875.

86. **Laguerre.** Sur la fonction exponentielle. Bull. de la Soc. Math. de France, vol. 8 (1880), pp. 11–18, or Oeuvres, vol. 1, p. 336.

87. **Humbert.** (a) Sur une généralisation de la théorie des fractions continues algébriques. Bull. de la Soc. Math. de France, vol. 8, pp. 191–196 ; vol. 9, pp. 24–30, 1879–1881.

(b) Sur la fonction $(x-1)^a$. Ibid., vol. 9, pp. 56–58, 1880–81.

88. **Pincherle.** Sulla rappresentazione approssimata di una funzione mediante irrazionali quadratici. Rendiconti, R. Istituto Lombardo di Scienze e Lettere, ser. 2, vol. 23, pp. 373–376, 1890.

89. **Pincherle.** (a) Una nuova estensione delle funzioni sferiche. Memoirie della R. Accad. delle Scienze dell'Istituto di Bologna, ser. 5, vol. 1, pp. 337–370, 1890.

(a') Sulla generalizzazione delle funzioni sferiche. Bologna Rendiconti, 1891–92, pp. 31–34.

(b) Un sistema d'integrali ellittici considerati come funzioni dell'invariante assoluto. Atti della R. Accad. dei Lincei, ser. 4, vol. 7_1, pp. 74–80, 1891.

90. **Bortolotti.** (a) Sui sistemi ricorrenti del 3° ordine ed in particolare sui sistemi periodici. Rendiconti del Circolo di Palermo, vol. 5, pp. 129–151, 1891.

(b) Sulla generalizzazione delle frazioni continue algebriche periodiche. Ibid., vol. 6, pp. 1–13, 1892.

VI. Series of Polynomials (Näherungsnenner).

The series

$$\frac{1}{v-u} = \sum_{n=0}^{\infty} (2n+1) Q^{(n)}(v) P^{(n)}(u)$$

was first given by *Heine* in Crelle's Jour., vol. 42 (1851), p. 72. See also his Handbuch, vol. 1, pp. 78–79, 197–200. Among the numerous works relating to expansions in terms of *Kugelfunctionen erster und zweiter Gattung* may be mentioned :

91. **Bauer.** Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen. Jour. für Math., vol. 56, pp. 101–121, 1859.

92. **C. G. Neumann.** Ueber die Entwickelung einer Function mit imaginärem Argumente nach den Kugelfunctionen erster und zweiter Gattung, Halle, 1862.

93. **Thomé.** Ueber die Reihen welche nach Kugelfunctionen fortschreiten. Jour. für Math., vol. 66, pp. 337–343, 1866.

94. **Laurent.** Mémoire sur les fonctions de Legendre. Jour. de Math., ser. 3, vol. 1, pp. 373–398, 1875.

See the comments by Heine in vol. 2, pp. 155–157, also by Darboux and Laurent in the same vol., pp. 240, 420.

Numerous memoirs relate to series in terms of the polynomials arising from the expansion of $(1 - 2ax + a^2)^{\nu}$. It suffices here to refer to the Encyklopädie der Math. Wissenschaften, I A 10, § 31.

---

95. **Frobenius.** Ueber die Entwicklung analytischer Functionen in Reihen, die nach gegebenen Functionen fortschreiten. Jour. für Math., vol. 73, pp. 1–30, 1871.

An interesting memoir.

96. **Darboux.** Sur l'approximation des fonctions de très-grands nombres et sur une classe étendue de développements en série, Part 2. Jour. de Math., ser. 3, vol. 4, pp. 377–416, 1878.

97. **Gegenbauer** Ueber Kettenbrüche. Wiener Berichte, vol. 80, Abth. 2, pp. 763–775, 1880.

98. **Poincaré.** (*a*) Sur les équations linéaires aux différentielles ordinaires et aux différences finies. Amer. Jour. of Math., vol. 7, pp. 243–257, 1885.

This gives an important criterion for the convergence of series of polynomials. See lecture 4.

(*a′*) Sur les séries des polynomes. Comp. Rend., vol. 56, p. 637, 1883.

99. On the series $\Sigma A_n(x - a_1)(x - a_2) \cdots (x - a_n)$.

A series of this form is employed in **Newton's** interpolation formula, Philosophiæ naturalis principia, book 3, lemma V. See the Encyklopädie der Math. Wissenschaften, I D 3, § 3. A similar use is made by

Cauchy. (*a*) Sur les fonctions interpolaires. Comp. Rend., vol. 11, pp. 775–789, 1841.

See next No. 95.

Peano. (*b*) Sulle funzioni interpolari. Atti della R. Accad. delle Scienze di Torino, vol. 18, pp. 573–580, 1883.

Bendixson. (*c*) Sur une extension à l'infini de la formule d'interpolation de Gauss. Acta Math., vol. 9, pp. 1–34, 1886.

(*c′*) Sur la formule d'interpolation de Lagrange. Comp. Rend., vol. 101 (1885), pp. 1050–1053 and 1129–1131.

Pincherle. (*d*) Sull'interpolazione. Memoirie della R. Accad. delle Scienze di Bologna, ser. 5, vol. 3, pp. 293–318.

(See a "note historique" by **Eneström**, Comp. Rend., vol. 103, p. 523, 1886).

See also No. 103.

100. **Pincherle,** Sur le développement d'une fonction analytique en série de polynomes. Comp. Rend., vol. 107, p. 986, 1888.

101. **Pincherle.** Résumé de quelques résultats relatifs à la théorie des systèmes recurrents de fonctions. Mathematical Papers, Chicago Congress, 1893, pp. 278–287.

102. **Blumenthal.** Ueber die Entwickelung einer willkürlichen Funktion nach den Nennern des Kettenbruches fur

$$\int_{-\infty}^{0} \frac{\phi(\xi)d\xi}{z-\xi}.$$

Dissertation, Göttingen, 1898.

The most advanced development of this subject is found in the work of *Blumenthal* and *Pincherle.*

103. **Laurent.** Sur les séries de polynomes. Jour. de Math., ser. 5, vol. 8, pp. 309–328, 1902.

104. **Stekloff.** Sur le développement d'une fonction donée en séries procédant suivant les polynomes de Tchébicheff et, en particulier, suivant les polynomes de Jacobi. Jour. für Math., vol. 125, pp. 207–236, 1903.

See also Nos. 20, 70, 71.

104 *bis.* **Rouché.** Mémoire sur le développement des fonctions en séries ordonnées suivant les dénominateurs des réduites d'une fraction continue. Jour. de l'Ec Polytech., cahier 37, pp. 1–34.

This memoir has a close connection with the work of Tchebychef.

VII. On the Roots of the Numerators and Denominators of the Convergents.

105. **Sylvester.** (*a*) On a remarkable modification of Sturm's theorem. Phil. Mag., ser. 4, vol. 5, pp. 446–456, 1853.

(*b*) Note on a remarkable modification of Sturm's theorem and on a new rule for finding superior and inferior limits to the roots of an equation. Ibid., vol. 6, pp. 14–20, 1853.

(*c*) On a new rule for finding superior and inferior limits to the real roots of any algebraic equation. Ibid., vol. 6, pp. 138–140, 1853.

(*d*) Note on the new rule of limits. Ibid., vol. 6, pp. 210–213, 1853.

(*e*) On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraic common measure. Phil. Trans., 1853; see in particular p. 496 ff.

(*f*) Théorème sur les limites des racines réelles des équations algébriques. Nouvelles Ann. de Math., ser. 1, vol. 12, pp. 286–287, 1853.

(*g*) Pour trouver une limite supérieure et une limite inférieure des racines réelles d'une équation quelconque. Ibid., ser. 1, vol. 12, pp. 329–336, 1853.

106. **Laguerre.** Sur quelques propriétés des équations algébriques qui ont toutes les racines réelles. Nouvelles Ann. de Math., ser. 2, vol. 19 (1880), pp. 224–239, or Oeuvres, vol. 1, pp. 113–118.

Laguerre considers here the roots of the numerators and denominators of the approximants for $f(x)$ and $1/f(x)$ when $f(x)$ is a polynomial with real roots.

107. **Gegenbauer.** (*a*) Ueber algebraische Gleichungen welche nur reele Wurzeln besitzen. Wiener Berichte, vol. 84 (1882), Abt. 2, see in particular pp. 1106–1107.

(*b*) Ueber algebraische Gleichungen welche eine bestimmte Anzahl complexer Wulzeln besitzen. Ibid., vol. 87, pp. 264–270, 1883.

108. **Markoff.** Sur les racines de certaines équations. Math. Ann., vol. 27, pp. 143–150, 1886.

108 *bis.* **Hurwitz.** Ueber die Nullstellen der Bessel'schen Function. Math. Ann., vol. 33, pp. 246–266, 1889.

Although the functions considered in this memoir are of a special character, the memoir is mentioned here on account of the methods employed.

109. **Porter.** On the roots of functions connected by a linear recurrent relation of the second order. Annals of Math., ser. 2, vol. 3, pp. 55–70, 1902.

See also Nos. 20, 26*a*, 31, 32*a*, 45, 56, 71, 74, 76, 87*a*, 118*a*.

VIII. Approximation to a Function at More Than One Point. Connection of Continued Fractions with the Theory of Interpolation.

Under No. 99 have been already classified various works which relate to simultaneous approximation at several points. In addition, the following memoirs may also be consulted:

110. **Cauchy.** Sur la formule de Lagrange relativ à interpolation. Analyse Alg., p. 528, or Oeuvres, ser. 2, vol. 3, pp. 429–433.

111. **Jacobi.** Ueber die Darstellung eine Reihe gegebner Werthe durch eine gebrochene rationale Function. Jour. für Math., vol. 30, pp. 127–156, 1846, or Werke, vol. 3, p. 479.

112. **Padé.** Sur l'extension des propriétés des réduites d'une fonction aux fractions d'interpolation de Cauchy. Comp. Rend., vol. 130, p. 697, 1900.

See also Nos. 95, 99.

For general works upon interpolation which bring out the relation of the subject to continued fractions, see Heine's Handbuch der Kugelfunctionen, vol. 2, and Markoff's Differenzenrechnung (deutsche Uebersetzung), chap. 1, 6, 7 ; also the following memoir :

113. **Posse.** Sur quelques applications des fractions continues algébriques. Pp. 1–175, 1886.

114. **Gauss.** Methodus nova integralium valores per approximationem inveniendi. Werke, vol. 3, pp. 165–196, 1816.

115. **Christoffel.** Ueber die Gaussische Quadratur und eine Verallgemeinerung derselben. Jour. für Math., vol. 55, pp. 61–82, 1858.

116. **Mehler.** Bemerkungen zur Theorie der mechanischen Quadraturen. Ibid., vol. 63, pp. 152–157, 1864.

117. **Posse.** Sur les quadratures. Nouvelles Ann. de Math., ser. 2, vol. 14, pp. 49–62, 1875.

118. **Stieltjes.** (*a*) Quelques recherches sur la théorie des quadratures dites mécaniques. Ann. de l'Ec. Nor., ser. 3, vol. 1, pp. 409–426, 1884.

We find here the origin of his notable 1894 memoir, No. 26*a*.

(*a′*) Sur l'évaluation approchée des intégrales. Comp. Rend., vol. 97, pp. 740 und 798, 1883.

(*b*) Note sur l' intégrale $\int_a^b f(x)G(x)dx$.

Nouv. Ann. de Math., ser. 3, vol. 7, pp. 161–171, 1888.

119. **Markoff.** Sur la méthode de Gauss pour le calcul approché des intégrales. Math. Ann., vol. 25, pp. 427–432, 1885.

120. **Pincherle.** Su alcune forme approssimate per la rappresentazione di funzioni. Memoirie della R. Accad. delle Scienze dell'Istituto di Bologna, ser. 4, vol. 10, pp. 77–88, 1889.

121. **Tchebychef.** A brief sketch of the memoirs below will be found on pp. 17–20 of *Vassilief's* memoir on "P. L. Tchebychef et son oeuvre scientifique."

(*a*) Sur les fractions continues. Jour. de Math., ser. 2, vol. 3, pp. 289–323, 1858, or Oeuvres, vol. 1, p. 203–230.

(*b*) Sur une formule d'analyse. Bull. Phys. Math. de l'Acad. des sciences de St. Pétersbourg, vol. 13, pp. 210–211, 1854, or Oeuvres, vol. 1, pp. 701–702.

(*c*) Sur une nouvelle série. Ibid., vol. 17, pp. 257–261, 1858, or Oeuvres, vol. 1, pp. 381–384.

(*d*) Sur l'interpolation par la méthode des moindres carrés. Mém. de l'Acad. des sciences de St. Pétersbourg, ser. 7, vol. 1, pp. 1–24, 1859, or Oeuvres, vol. 1, pp. 473–498.

(*e*) Sur le développement des fonctions à une seule variable. Bull. de l'Acad. imp. des sciences de St. Pétersbourg, ser. 7, vol. 1, pp. 194–199, 1860, or Oeuvres, vol. 1, pp. 501–508.

IX. Miscellaneous.

122. **Tchebychef.** (*a*) Sur les fractions continues algébriques. Jour. de Math., ser. 2, vol. 10, pp. 353–358, 1865, or Oeuvres, vol. 1, pp. 611–614.

(*b*) Sur le développement des fonctions en séries à l'aide des fractions continues, 1866. Oeuvres, vol. 1, pp. 617–636.

(*c*) Sur les expressions approchées, linéares par rapport a deux polynomes. Bull. des sciences Math. et Astron., ser. 2, vol. 1, pp. 289, 382; 1877.

Hermite. (*d*) Sur une extension donnée à la théorie des fractions continues par M. Tchebychef. Jour. für Math., vol. 88, pp. 12–13, 1880.

123. **Tchebychef.** (*a*) Sur les valeurs limites des intégrales. Jour. de Math., ser. 2, vol. 19, pp. 157–160, 1874.

(*b*) Sur la representation des valeurs limites des intégrales par des residus integraux (1885). Acta. Math. vol. 9, pp. 35–56, 1887.

Markoff. (*c*) Démonstration de certaines inégalités de M. Tchebychef. Math. Ann., vol. 24, pp. 172–178, 1884.

(*d*) Nouvelles applications des fractions continues. Math. Ann., vol. 47, pp. 579–597, 1896.

124. **Laguerre.** (*a*) Sur le développement de $(x-z)^m$ suivant les puissances de (z^2-1). Comp. Rend., vol. 86 (1878), p. 956, or Oeuvres, vol. 1, p. 315.

(*b*) Sur le développement d'une fonction suivant les puissances d'une polynome. Jour. für Math., vol. 88 (1880); in particular, p. 37, or Oeuvres, vol. 1, p. 298.

(*c*) Same subject. Comp. Rend., vol. 86, (1878) p. 383, or Oeuvres, vol. 1, p. 295.

(*d*) Sur quelques théorémes de M. Hermite. Extrait d'une lettre addressée à M. Borchardt. Jour. für Math., vol. 89 (1880), pp. 340–342, or Oeuvres, vol. 1, p. 360.

125. **Sylvester.** Preuve que π ne peut pas être racine d'une équation algébrique à coefficients entiers. Comp. Rend., vol. 111, pp. 866–871, 1890.

A fundamental error in the proof has been pointed out by Markoff. See p. 386 of vol. 30 of the Fortschritte der Math.

126. **Gegenbauer.** Ueber die Näherungsnenner regulärer Kettenbrüche. Monatshefte für Math. und Phys., vol. 6, pp. 209–219, 1895.

127. **Bortolotti.** Sulla rappresentazione approssimata di funzioni algebriche per mezzo di funzioni razionale. Atti della R. Accad. dei Lincei, ser. 5, vol. 1_1, pp. 57–64, 1899.

Addendum to I A.

128. **Euler.** De fractionibus continuis dissertatio. Comment. Petrop., vol. 9, p. 129 ff., 1737.

[redacted]